GEOMETRY OF GRAPHS OF DISCS IN A HANDLEBODY

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ABSTRACT. For a handlebody H of genus $g \geq 2$, we investigate three distinct graph of discs and show that they are hyperbolic. One of these graphs is the electrified disc graph whose vertices are isotopy classes of essential discs in H. Two discs D_1, D_2 are connected by an edge of length one if and only if there is an essential simple closed curve on ∂H which can be realized disjointly from both D_1, D_2. This is used to give an alternative proof of hyperbolicity of the usual disc graph. We also observe that there is a coarsely surjective Lipschitz projection of the electrified disc graph onto the free factor graph of the free group with n generators.

1. INTRODUCTION

The *curve graph* $\mathcal{CG}$ of a closed oriented surface S of genus $g \geq 2$ is the graph whose vertices are isotopy classes of essential (i.e. non-contractible) simple closed curves on S and where two such curves are connected by an edge of length one if and only if they can be realized disjointly. The curve graph is a hyperbolic geodesic metric space of infinite diameter [MM99]. The *mapping class group* $\mathrm{Mod}(S)$ of all isotopy classes of orientation preserving homeomorphisms of S acts on $\mathcal{CG}$ as a group of simplicial isometries. This action is *coarsely transitive*, i.e. there are only finitely many orbits of vertices. The curve graph turned out to be an important tool for understanding the geometry of the mapping class group of S [MM00].

For mapping class groups of other manifolds, much less is known. Examples of manifolds with large and interesting mapping class groups are *handlebodies*. A handlebody of genus $g \geq 2$ is a compact three-dimensional manifold H which can be realized as a closed regular neighborhood in $\mathbb{R}^3$ of an embedded bouquet of g circles. Its boundary ∂H is a closed oriented surface of genus g. The group $\mathrm{Map}(H)$ of all isotopy classes of orientation preserving homeomorphisms of H is called the *handlebody group* of H. The homomorphism which associates to an orientation preserving homeomorphism of H its restriction to ∂H induces an embedding of $\mathrm{Map}(H)$ into $\mathrm{Mod}(\partial H)$.

For a number $L > 1$, a map Φ of a metric space X into a metric space Y is called an *L-quasi-isometric embedding* if

$$d(x,y)/L - L \leq d(\Phi x, \Phi y) \leq Ld(x,y) + L \text{ for all } x, y \in X.$$

The inclusion $\mathrm{Map}(H) \to \mathrm{Mod}(\partial H)$ in *not* a quasi-isometric embedding [HH11] and hence understanding the geometry of the mapping class group does not lead to an understanding of the geometry of $\mathrm{Map}(H)$.

Instead one can try to develop tools for the investigation of $\mathrm{Map}(H)$ which are similar to the tools for the mapping class group. In particular, there is an analog of the curve graph for a handlebody which is defined as follows.

An *essential disc* in H is a properly embedded disc $(D, \partial D) \subset (H, \partial H)$ which is not homotopic into ∂H. This is equivalent to stating that the boundary ∂D of D is an essential simple closed curve in ∂H.

Definition 1. The *disc graph* $\mathcal{DG}$ of H is the graph whose vertices are isotopy classes of essential discs in H. Two such discs are connected by an edge of length one if and only if they can be realized disjointly.

The handlebody group acts as a group of simplicial isometries on the disc graph, and this action is coarsely transitive.

Associating to an essential disc its boundary defines a simplicial embedding of the disc graph of H into the curve graph of ∂H. Its image is a quasiconvex subspace of the curve graph [MM04]. However, the curve graph and the disc graph are not locally finite, and the embedding $\mathcal{DG} \to \mathcal{CG}$ is *not* a quasi-isometric embedding. Nevertheless, Masur and Schleimer showed [MS10]

Theorem 1. *The disc graph is hyperbolic.*

The first goal of this note is to construct a graph whose vertices are discs in H, which admits a coarsely transitive action of $\mathrm{Map}(H)$ as a group of isometries and such that the vertex inclusion into the curve graph is a quasi-isometric embedding.

To this end, call a simple closed curve c on ∂H *discbusting* if c has an essential intersection with the boundary of every disc. Define a *discbusting I-bundle* to be a simple closed curve c on ∂H with the following property. There is a compact surface F with connected boundary ∂F and there is a homeomorphism of the orientable I-bundle over F onto H which maps ∂F to c. We say that F is the *base surface* of the discbusting I-bundle c. There is an orientation reversing involution $\iota_c : \partial H \to \partial H$ whose fixed point set is the curve c and such that $\partial H / \iota_c$ can naturally be identified with the base surface F. The handlebody group preserves the set of discbusting I-bundles.

Definition 2. The *super-conducting disc graph* is the graph $\mathcal{SDG}$ whose vertices are isotopy classes of essential discs in H and where two vertices D_1, D_2 are connected by an edge of length one if and only if one of the following two possibilities holds.

(1) There is a simple closed curve on ∂H which can be realized disjointly from both $\partial D_1, \partial D_2$.
(2) There is a discbusting I-bundle c which intersects both $\partial D_1, \partial D_2$ in precisely two points.

Since the distance in the curve graph of ∂H between two curves which intersect in two points does not exceed 3 [MM99], the vertex-inclusion extends to a 6-Lipschitz map $\mathcal{SDG} \to \mathcal{CG}$. We show

Theorem 2. *The vertex inclusion defines a quasi-isometric embedding $\mathcal{SDG} \to \mathcal{CG}$. In particular, $\mathcal{SDG}$ is hyperbolic of infinite diameter.*

There is another topologically meaningful $\mathrm{Map}(H)$-graph whose vertices are discs. This graph is defined as follows.

Definition 3. The *electrified disc graph* is the graph $\mathcal{EDG}$ whose vertices are isotopy classes of essential discs in H and where two vertices D_1, D_2 are connected by

an edge of length one if and only if there is an essential simple closed curve on ∂H which can be realized disjointly from both $\partial D_1, \partial D_2$.

Since for any two disjoint simple closed curves c, d on ∂H there is a simple closed curve on ∂H which can be realized disjointly from c, d (e.g. one of the curves c, d), the electrified disc graph is obtained from the disc graph by adding some edges. We use Theorem 2 to show

Theorem 3. *The electrified disc graph is hyperbolic.*

The vertex inclusion $\mathcal{EDG} \to \mathcal{CG}$ is not a quasi-isometric embedding. However, we can estimate distances in $\mathcal{EDG}$ as follows.

By an *arc* in a surface X with connected boundary ∂X we mean a homotopy class relative to ∂X of an embedded arc in X with both endpoints in ∂X. For a discbusting I-bundle c define the *arc and disc graph* $\mathcal{C}'(c)$ of c to be the graph whose vertices are essential simple closed curves in the base surface F or arcs with both endpoints on the boundary of F and where two such arcs or discs are connected by an edge of length one if and only if they can be realized disjointly.

Every disc $D \subset H$ intersects c. Moreover, up to isotopy its boundary is invariant under the involution ι_c (see the discussion in [MS10]) and hence it defines a collection of pairwise disjoint arcs on the surface F. Define $\pi^c(\partial D)$ to be the union of these arcs viewed as points in $\mathcal{C}'(c)$. For a collection A of discs in H we then write $\operatorname{diam}(\pi^c(A))$ to denote the diameter of $\pi^c(A)$ in the arc and disc graph $\mathcal{C}'(c)$.

For a number $C > 0$ and a subset X of $\mathcal{C}'(c)$ let $\operatorname{diam}(X)_C$ be the diameter of X if this diameter is bigger than C and let $\operatorname{diam}(X)_C = 0$ otherwise. Let $d_{\mathcal{E}}$ be the distance in the electrified disc graph of H, and let $d_{\mathcal{CG}}$ be the distance in the curve graph of ∂H. We obtain the following estimate for $d_{\mathcal{E}}$.

Theorem 4. *There is a number $C > 0$ such that*

$$d_{\mathcal{E}}(D, E) \asymp d_{\mathcal{CG}}(\partial D, \partial E) + \sum_c \operatorname{diam}(\pi^c(\partial D \cup \partial E))_C$$

where the sum is over all discbusting I-bundles in ∂H and $\asymp$ means equality up to a fixed multiplicative and additive constant.

We use Theorem 3 to give an independent proof of Theorem 1. In analogy to the results of [MS10] we also obtain a distance formula for the disc graph $\mathcal{DG}$ formulated in Corollary 7.4.

A hyperbolic geodesic metric space admits a *Gromov boundary*. The Gromov boundary of the curve graph of ∂H can be described as follows. The space of *geodesic laminations* on ∂H can be equipped with the *coarse Hausdorff topology*. In this topology, a sequence (λ_i) of geodesic laminations converges to a geodesic lamination λ if every accumulation point of (λ_i) in the usual Hausdorff topology contains λ as a sublamination. This topology is not T_0, but its restriction to the subset $\partial \mathcal{CG}$ of all *minimal geodesic laminations* which fill up ∂H (i.e. which have an essential intersection with every simple closed curve in ∂H) is Hausdorff. The Gromov boundary of the curve graph can naturally be identified with the space $\partial \mathcal{CG}$ equipped with the coarse Hausdorff topology [H06] (see also the references there).

Let

$$\partial \mathcal{SDG} \subset \partial \mathcal{CG}$$

be the closed subset of all points which are a limit in the coarse Hausdorff topology of a sequence of boundaries of discs.

For every discbusting I-bundle c let $\partial\mathcal{E}(c)$ be the set of all geodesic laminations which fill up $\partial H - c$ and which are limits in the coarse Hausdorff topology of boundaries of discs intersecting c in precisely two points. Define

$$\partial\mathcal{EDG} = \partial\mathcal{SCG} \cup \bigcup_c \partial\mathcal{E}(c)$$

where the union is over all discbusting I-bundles.

Call a simple closed non-separating curve c on ∂H *visible* if c is not discbounding and if there are two discs D, E whose boundary fill up $\partial H - c$. If c is visible then the disc graph $\mathcal{DG}(c)$ of $\partial H - c$ is defined. For every visible simple closed curve c on ∂H let $\partial\mathcal{DG}(c)$ be the boundary of the disc graph of $\partial H - c$. Recursively, $\partial\mathcal{DG}(c)$ can be viewed as a subspace of measured lamination space equipped with the coarse Hausdorff topology. Define

$$\partial\mathcal{DG} = \partial\mathcal{EDG} \cup \bigcup_c \partial\mathcal{DG}(c)$$

where as before, $\partial\mathcal{DG}$ is equipped with the coarse Hausdorff topology. We have

Theorem 5. *The Gromov boundary of $\mathcal{SDG}$ (or of $\mathcal{EDG}$ or of $\mathcal{DG}$) can be identified with $\partial\mathcal{SDG}$ (or with $\partial\mathcal{EDG}$ or with $\partial\mathcal{DG}$).*

One can also study graphs of discs in a handlebody with a finite collection of marked points on the boundary. The disc graph of a handlebody with a marked point p is the graph whose vertices are essential discs in H with boundary in $\partial H - p$. Two such vertices are connected by an edge of length one if and only if they can be realized disjointly. However, we observe in Section 8 that the disc graph of a handlebody of genus $g = 2m \geq 2$ with one marked point on the boundary is not hyperbolic. We believe that the same holds true for an arbitrary handlebody of genus at least two with marked points, but we did not check this. In contrast, the disc graph of a solid torus with marked points on the boundary is hyperbolic [KLS09].

One of the motiviations for introducing the electrified disc graph comes from an attempt to find an analog of the curve graph for the free group F_g of rank $g \geq 3$ which is a hyperbolic geodesic metric graph, with a coarsely transitive action of the outer automorphism group $\mathrm{Out}(F_g)$ as a group of isometries. One candidate is the *free factor graph* $\mathcal{FF}$ of F_g which is defined to be the graph whose vertices are conjugacy classes of free factors of F_g. Two such conjugacy classes of free factors are connected by an edge of length one if up to conjugation the factors have a common refinement [KL09].

The action of $\mathrm{Map}(H)$ on conjugacy classes of the fundamental group F_g of H defines a surjective homomorphism $\rho : \mathrm{Map}(H) \to \mathrm{Out}(F_g)$. With this terminology, the electrified disc graph of H is related to the free factor graph of F_g as follows.

Proposition. *There is a ρ-equivariant Lipschitz map $\mathcal{EDG} \to \mathcal{FF}$.*

To be more precise, we observe that the free factor graph is bilipschitz equivalent to a graph constructed from spheres in complete analogy to the definition of the electrified disc graph.

There are other graphs on which $\mathrm{Out}(F_g)$ acts coarsely transitively as a group of simplicial isometries [KL09]. One of these graphs is the sphere graph constructed

from spheres in analogy to the definition of the disc graph. A third such graph is the *graph of free splittings* which corresponds to the graph of separating spheres. This graph is not hyperbolic [SS10].

The organization of this note is as follows. In Section 2 we use surgery of discs to relate distance and intersection in the superconducting disc graph of H. In Section 3 we estimate the distance in the curve graph using train tracks. This together with a construction of [MM04] is used in Section 4 to show Theorem 2 and the first part of Theorem 5. In Section 5 we establish a condition for hyperbolicity of a graph which can be obtained from a hyperbolic graph by deleting some edges; this condition is motivated by the work of Farb [F98]. In Section 6, we deduce Theorem 3, Theorem 4 and the second part of Theorem 5 from Theorem 2. Section 7 contains an alternative proof of Theorem 1 as well as the proof of the third part of Theorem 5. Along the way we construct $3g - 5$ additional hyperbolic metric Map(H)-graphs whose vertex sets are the set of all discs in H. These graphs lie geometrically between $\mathcal{EDG}$ and the disc graph $\mathcal{DG}$, i.e. they can be obtained from $\mathcal{DG}$ by adding edges and from $\mathcal{EDG}$ by deleting edges (with a slight abuse of notation). In Section 8 we show that the disc graph of a handlebody with marked point is in general not hyperbolic. Section 9 is devolted to explaining the relation of the electrified disc graph and the free factor graph for Out(F_g).

2. Distance and intersection

In this section we consider a handlebody H of genus $g \geq 2$. We use surgery of discs to establish first estimates for distances in the electrified disc graph $\mathcal{EDG}$ and the superconducting disc graph $\mathcal{SDG}$ of H. We begin with introducing the basic surgery construction needed later on.

By a disc in the handlebody H we always mean an essential disc in H. Two discs D_1, D_2 are in *normal position* if their boundary circles intersect in the minimal number of points and if every component of $D_1 \cap D_2$ is an embedded arc in $D_1 \cap D_2$ with endpoints in $\partial D_1 \cap \partial D_2$. In the sequel we always assume that discs are in normal position; this can be achieved by an isotopy of the discs.

Let D be any disc and let E be a disc which is not disjoint from D. An *outer arc* of ∂E relative to D is a component α of $\partial E - D$ which intersects ∂D in opposite directions at its endpoints. This means that α leaves and returns to the same side of $\partial D \subset \partial H$. We also require that there is a component E' of $E - D$ whose boundary is composed of α and an arc $\beta \subset D$. The interior of β is contained in the interior of D. We call such a disc E' an *outer component* of $E - D$.

For every disc E which is not disjoint from D there are at least two distinct outer components E', E'' of $E - D$. There may also be components of $\partial E - D$ which leave and return to the same side of D but which are not outer arcs. An example of such a component α is an arc contained in the boundary of a rectangle component

of $E - D$ which is bounded by two arcs contained in D and two subarcs of ∂E with endpoints on ∂D which are homotopic to α relative to ∂D.

Let $E' \subset E$ be an outer component of $E - D$ whose boundary is composed of an outer arc and a subarc $\beta = E' \cap D$ of D. The arc β decomposes the disc D into two half-discs P_1, P_2. The unions $Q_1 = E' \cup P_1$ and $Q_2 = E' \cup P_2$ are embedded discs in H which up to isotopy are disjoint and disjoint from D. We say that the disc Q_i is obtained from D by *simple surgery* at the outer component E' of $E - D$ (see e.g. [S00] for this construction). The discs Q_1, Q_2 are essential. If γ is a simple closed curve on ∂H which is disjoint from $D \cup E$ then γ is also disjoint from each of the discs Q_i.

Each disc in H can be viewed as a vertex in the disc graph $\mathcal{DG}$, the electrified disc graph $\mathcal{EDG}$ and the superconducting disc graph $\mathcal{SDG}$. We will work with all three graphs simultaneously. To this end we denote by $d_\mathcal{D}$ (or $d_\mathcal{E}$ or $d_\mathcal{S}$) the distance in $\mathcal{DG}$ (or in $\mathcal{EDG}$ or in $\mathcal{SDG}$). Note that for any two discs D, E we have

$$d_\mathcal{S}(D, E) \leq d_\mathcal{E}(D, E) \leq d_\mathcal{D}(D, E).$$

In the sequel we always assume that all curves and multicurves on ∂H are essential. For two simple closed multicurves c, d on ∂H let $\iota(c, d)$ be the geometric intersection number between c, d. The following lemma is contained in [MM04]. We provide the short proof for completeness.

Lemma 2.1. *Let D_1, D_2 be two discs. Then D_1 can be connected to a disc D_2' which is disjoint from D_2 by at most $\iota(\partial D_1, \partial D_2)$ simple surgeries. In particular,*

$$d_\mathcal{D}(D_1, D_2) \leq \iota(\partial D_1, \partial D_2) + 1.$$

Proof. Let D_1, D_2 be two discs in normal position. Assume that D_1, D_2 are not disjoint. Then there is an outer component of $D_1 - D_2$. There is an essential disc D' obtained from simple surgery of D_2 at this component which is disjoint from D_2 and such that

$$(1) \qquad\qquad\qquad \iota(\partial D_1, \partial D') < \iota(\partial D_1, \partial D_2).$$

The lemma now follows by induction on $\iota(\partial D_1, \partial D_2)$. $\qquad\qquad\square$

We will also use relative versions of the superconducting disc graph and the electrified disc graph. For this we first introduce some more terminology (compare [S00, MS10]). Namely, a simple closed multicurve β on ∂H is called *discbusting* if β intersects the boundary of every disc in H. A *discbusting I-bundle* is a simple closed curve $\gamma \subset \partial H$ with the following property. There is an oriented I-bundle over a surface F with connected boundary ∂F, and there is a homeomorphism of this I-bundle onto H which maps ∂F to γ.

If β is a simple multicurve which is not discbusting and without discbounding components then a *discbusting I-bundle in $\partial H - \beta$* is a simple closed curve $\gamma \subset \partial H - \beta$ with the following property. There is a surface F with boundary ∂F and there is a homeomorphism of an oriented I-bundle over F onto H which maps the boundary of F onto $\gamma \cup \beta$. Note that ∂F is not connected and γ is *not* discbusting.

An example can be obtained as follows. Let F be an orientable surface of genus k with two boundary components. The oriented I-bundle over F is homeomorphic to a handlebody H of genus $2k + 1$. A boundary component β of F is not discbusting in H. The second boundary component of F defines a discbusting I-bundle in $\partial H - \beta$.

Let $\mathcal{SDG}(\beta)$ be the graph whose vertices are discs which do not intersect β and where two such discs D, E are connected by an edge of length one if one of the following three possibilites is satisfied.

(1) D, E are disjoint.
(2) There is an essential simple closed curve $\alpha \subset \partial H - \beta$ (i.e. which is essential as a curve in the possibly disconnected subsurface $\partial H - \beta$ of ∂H) which is disjoint from $D \cup E$.
(3) There is discbusting I-bundle $\gamma \subset \partial H - \beta$ which intersects both D, E in precisely two points.

Note that $\mathcal{SDG}(\beta)$ is a subgraph of $\mathcal{SDG}$. Since surgery of two discs disjoint from a multicurve β on ∂H yields a disc disjoint from β, Lemma 2.1 and its proof shows that the graph $\mathcal{SCG}(\beta)$ is connected.

Similarly, for a simple closed multicurve β which is not discbusting and which does not contain a discbounding component we define $\mathcal{EDG}(\beta)$ to be the graph whose vertices are discs which do not intersect β and where two such discs D, E are connected by an edge of length one if and only if either D, E are disjoint or if there is an essential simple closed curve γ in $\partial H - \beta$ which can be realized disjointly from both D, E.

For a simple closed multicurve β denote by $d_{\mathcal{E}, \beta}$ the distance in $\mathcal{EDG}(\beta)$.

Lemma 2.2. *Let $D, E \subset H$ be essential discs. If there is a simple closed curve $\alpha \subset \partial H$ which intersects ∂E in at most one point and which intersects ∂D in at most $k \geq 1$ points then $d_{\mathcal{E}}(D, E) \leq \log_2 k + 3$. If moreover $\partial D, \partial E, \alpha$ are disjoint from a simple closed multicurve $\beta \subset \partial H$ then $d_{\mathcal{E}, \beta}(D, E) \leq \log_2 k + 3$.*

Proof. Let $D, E \subset H$ be essential discs in normal position as in the lemma. If D, E are disjoint then there is nothing to show, so assume that $\partial D \cap \partial E \neq \emptyset$. Let $\alpha \subset \partial H$ be a simple closed curve which intersects ∂E in at most one point and which intersects ∂D in $k \geq 0$ points. If α is disjoint from both D, E then $d_{\mathcal{E}}(D, E) \leq 1$ by definition of the electrified disc graph. Thus we may assume that $k \geq 1$. Via a small homotopy we may moreover assume that α is disjoint from $D \cap E$.

We modify D as follows. There are at least two outer components of $E - D$. Since α intersects ∂E in at most one point, one of these components, say the component E', is disjoint from α. The boundary of E' decomposes D into two subdiscs P_1, P_2. Assume without loss of generality that P_1 contains fewer intersection points with α than P_2. Then P_1 intersects α in at most $k/2$ points. The disc $D' = P_1 \cup E'$ has at most $k/2$ intersection points with α, and up to isotopy, it is disjoint from D. In particular, we have $d_{\mathcal{E}}(D, D') = 1$.

Repeat this construction with D', E. After at most $\log_2 k + 1$ such steps we obtain a disc D_1 which either is disjoint from E or is disjoint from α. The distance in the disc graph $\mathcal{DG}$ between D and D_1 is at most $\log_2 k + 1$.

If D_1 and E are disjoint then $d_{\mathcal{D}}(D_1, E) \leq 1$ and $d_{\mathcal{D}}(D, E) \leq \log_2 k + 2$ and we are done. Otherwise apply the above construction to D_1, E but with the roles of D_1 and E exchanged. We obtain a disc E_1 which is disjoint from both E and α, in particular it satisfies $d_{\mathcal{D}}(E_1, E) = 1$. The discs D_1, E_1 are both disjoint from α and therefore $d_{\mathcal{E}}(D_1, E_1) \leq 1$ by the definition of the electrified disc graph. Together this shows that $d_{\mathcal{E}}(D, E) \leq \log_2 k + 1 + d_{\mathcal{E}}(D_1, E) \leq \log_2 k + 3$.

If D, E, α are disjoint from the simple closed multicurve β then the same holds true for each of the surgered discs and we conclude that $d_{\mathcal{E},\beta}(D, E) \leq \log_2 k + 3$. $\square$

Remark: The proof of Lemma 2.2 also shows that a discbusting curve intersects the boundary of every disc in at least two points.

If $D, E \subset H$ are discs in normal position then each component of $D - E$ is a disc and therefore the graph dual to the cell decomposition of D whose two-cells are the components of $D - E$ is a tree. If $D - E$ only has two outer components then this tree is just a line sement. The following lemma analyzes the case that this holds true for both $D - E$ and $E - D$.

Lemma 2.3. *Let $D, E \subset H$ be discs in normal position. If $D - E$ and $E - D$ only have two outer components then one of the following two possibilities are satisfied..*

(1) $d_{\mathcal{E}}(D, E) \leq 4$.
(2) D, E *intersect some discbusting I-bundle γ in precisely two points.*

If D, E are disjoint from a simple closed multicurve β then the same alternative holds true for D, E viewed as discs with boundary in $\partial H - \beta$.

Proof. Let D, E be two discs in normal position such that $D - E$ and $E - D$ only have two outer components. Then each component of $D - E, E - D$ either is an outer component or a rectangle, i.e. a disc whose boundary consists of two components of $D \cap E$ and two arcs contained in ∂H. Since $d_{\mathcal{E}}(D, E) = 1$ if there is a simple closed curve in ∂H which is disjoint from $\partial D \cup \partial E$, we may assume without loss of generality that $\partial D \cup \partial E$ fills up ∂H (or $\partial H - \beta$ if D, E are disjoint from the simple closed multicurve β without discbounding components).

Choose tubular neighborhoods $N(D), N(E)$ of D, E in H which are homeomorphic to an interval bundle over a disc. Assume that $N(D), N(E)$ intersects ∂H in an embedded annulus whose interior we denote by $A(D), A(E)$. Then $\partial N(D) - A(D)$, $\partial N(E) - A(E)$ is the union of two disjoint discs isotopic to D, E. We may assume that $\partial N(D) - A(D)$ is in normal position with respect to $\partial N(E) - A(E)$ and that $S = \partial(N(D) \cup N(E)) - (A(D) \cup A(E))$ is a compact surface with boundary which is properly embedded in H. Since H is assumed to be oriented, the boundary $\partial(N(D) \cup N(E))$ has an induced orientation which restricts to an orientation of S. If $\beta \subset \partial H$ is a simple closed multicurve disjoint from $D \cup E$ then we may assume that $N(D) \cup N(E)$ is disjoint from β as well.

Let (P, σ) be a pair consisting of an outer component P of $D - E$ and an orientation σ of D. The orientation σ determines a side of D in $N(D)$, say the right side. We claim that the component Q of the surface S containing the copy of P in $\partial N(D)$ to the right of D is a disc which contains precisely one other pair (P', σ') of this form.

Namely, by construction, each component of $\partial N(D) - (A(D) \cup N(E))$ either corresponds to an outer component of $D - E$ and the choice of a side, or it corresponds to a rectangle component of $D - E$ and a choice of a side. A component corresponding to a rectangle is glued at each of its two sides which are contained in the interior of H to a component of $\partial N(E) - (A(E) \cup N(D))$. In other words, up to homotopy, the component Q of S can be written as a chain of oriented discs beginning with P and alternating between components of $D - E$ and $E - D$ equipped with one of the two possible orientations. Since Q is embedded in H and contains

P, this chain can not be a cycle and hence it has to terminate at an oriented outer component of $D - E$ or $E - D$ which is distinct from (P, σ).

To summarize, for each pair (P, σ) consisting of an outer component P of $D - E$ or $E - D$ and an orientation σ of D or E there is a component of S which is a properly embedded disc in H. This disc is disjoint from $D \cup E$ and corresponds to precisely two such pairs (P, σ), so there is a total of four such discs. Denote these discs by $Q_1, \ldots, Q_4$. If one of these discs is essential, say if this holds true for the disc Q_i, then $d_{\mathcal{D}}(D, Q_i) \leq 1, d_{\mathcal{D}}(E, Q_i) \leq 1$ and we are done.

Otherwise define a *cycle* in $\{Q_1, \ldots, Q_4\}$ to be a subset C of minimal cardinality so that the following holds true. Let $Q_i \in C$ and assume that Q_i contains a pair (B, ζ) consisting of an outer component B of $D - E$ (or of $E - D$) and an orientation ζ of D (or E). If Q_j is the disc containing the pair (B, ζ') where ζ' is the orientation distinct from ζ then $Q_j \in C$. Note that two distinct cycles are disjoint. For each cycle C we construct a properly embedded annulus $A(C) \subset H$ as follows. Remove from each of the discs Q_i in the cycle the subdiscs which correspond to outer components of $D - E, E - D$ and glue the discs along the boundary arcs of these outer components.

To be more precise, let B be an outer component of $D - E$ corresponding to a subdisc of Q_i and let $\beta = B \cap E$. Then the complement of B in Q_i (with a small abuse of notation) contains the arc β in its boundary, and the orientation of Q_i defines an orientation of β. There is a second disc Q_j in the cycle which contains B and which induces on β the opposite orientation (Q_j is not necessarily distinct from Q_i). Glue Q_i to Q_j along β and note that the resulting surface is oriented. Doing this with each of the outer components of $D - E$ and $E - D$ contained in the cycle yields a properly embedded annulus $A(C) \subset H$ as claimed.

If there is a cycle C of odd length then a boundary curve γ of $A(C)$ intersects one of the two discs D, E, say the disc D, in precisely one point, and it intersects the disc E in at most two points. In particular, γ is essential. Moreover, Lemma 2.2 shows that $d_{\mathcal{E}}(D, E) \leq 4$.

Similarly, if there is a cycle C of length two then there are two possibilities. The first case is that the cycle contains both an outer component of $D - E$ and an outer component of $E - D$. Then a boundary curve γ of $A(C)$ intersects each of the discs D, E in precisely one point. In particular, γ is essential and $d_{\mathcal{E}}(D, E) \leq 3$.

If C contains both outer components of say the disc D then a boundary curve γ of $A(C)$ intersects D in precisely two points and it is disjoint from E. Let D' be a disc obtained from D by a disc exchange at an outer components of $E - D$. Then either D' is disjoint from both D, E (which is the case if D' is composed of an outer component of $E - D$ and an outer component of $D - E$) or D' intersects γ in precisely one point and is disjoint from D. As before, we conclude from Lemma 2.2 that $d_{\mathcal{E}}(D, E) \leq 4$.

We are left with the case that there is a single cycle C of length four. Let γ_1, γ_2 be the two boundary curves of $A(C)$. We claim that γ_1, γ_2 are freely homotopic. Namely, assume that the discs Q_i are numbered in such a way that Q_i and Q_{i+1} share one outer component of $D - E$ or $E - D$. Glue the discs $Q_1, \ldots, Q_4$ successively to a single disc Q with the surgery procedure described above (namely, glue Q_1 to Q_2 to form a disc $\hat{Q}_1$ and then glue $\hat{Q}_1$ to Q_3 etc). Since each of the discs Q_i is contractible, the same holds true for the disc Q. But this implies that the annulus

$A(C)$ is homotopic into the boundary of H and therefore its boundary curves are freely homotopic.

Assume from now on that $A(C) \subset \partial H$. Then $\partial D, \partial E$ intersects $A(C)$ in two arcs connecting the two boundary components of $A(C)$. Up to homotopy, $D \cup E = A(C) \cup \bigcup_i P_i \cup \bigcup_j R_j$ where for all i, j, P_i is a rectangle component of $D - E$ and R_j is a rectangle component of $E - D$. These rectangles meet at common boundary arcs. In other words, $D \cup E$ is an I-bundle embedded in H (compare p.31 of [MS10]). Since $\partial D \cup \partial E$ jointly fill up ∂H, each of the complementary components of $D \cup E$ is a ball. The union of these balls with $D \cup E$ has again the structure of an I-bundle. To summarize, $D \cup E$ defines the structure of an I-bundle for H with vertical boundary $A(C)$ and horizontal boundary $\partial H - A(C)$. A boundary component γ of $A(C)$ is discbusting.

Now if the discs D, E are disjoint from a simple closed multicurve $\beta \subset \partial H$ then the entire construction takes place in $\partial H - \beta$ and hence either $d_{\mathcal{E},\beta}(D, E) \leq 4$ or D, E intersect a discbusting I-bundle in $\partial H - \beta$ in precisely two points. $\square$

We use Lemma 2.3 to improve Lemma 2.2 as follows.

Proposition 2.4. *Let $D, E \subset H$ be essential discs. If there is a simple closed curve $\alpha \subset \partial H$ which intersects $\partial D, \partial E$ in at most $k \geq 1$ points then $d_{\mathcal{S}}(D, E) \leq 2k + 4 + 2^m$. If D, E are disjoint from a simple closed multicurve β then $d_{\mathcal{S},\beta}(D, E) \leq 2k + 4 + 2^m$.*

Proof. Let D, E be essential discs in normal position as in the lemma which are not disjoint.

Let α be a simple closed curve which intersects both ∂D and ∂E in at most $k \geq 1$ points. We may assume that these intersection points are disjoint from $\partial D \cap \partial E$.

Let $p \geq 2$ (or $q \geq 2$) be the number of outer components of $D - E$ (or of $E - D$). If $p = 2, q = 2$ then Lemma 2.3 shows that $d_{\mathcal{S}}(D, E) \leq 4$.

Let $j \leq k, j' \leq k$ be the number of intersection points of D, E with α. If $\min\{j, j'\} \leq 1$ then $d_{\mathcal{S}}(D, E) \leq \log_2 k + 3$ by Lemma 2.2. Thus it suffices to show the following. If $\max\{p, q\} \geq 3$ and if $\min\{j, j'\} \geq 2$ then there is a simple surgery transforming the pair (D, E) to a pair (D', E') (here either $D = D'$ or $E = E'$ and D is disjoint from D', E is disjoint from E') so that the total number of intersections of $D' \cup E'$ with α is strictly smaller than $j + j'$.

For this assume without loss of generality that $q \geq 3$. If $j/2 > j'/3$ then choose an outer component E_1 of $E - D$ with at most $j'/3$ intersections with α. This is possible because $E - D$ has at least three outer components. Let D_1 be a component of $D - E_1$ which intersects α in at most $j/2$ points. Then $D_1 \cup E_1$ is a disc which is disjoint from D and has at most $j/2 + j'/3 < j$ intersections with α.

On the other hand, if $j/2 \leq j'/3$ then choose an outer component D_1 of $D - E$ with at most $j/2$ intersections with α. Let E_1 be a component of $E - D_1$ with at most $j'/2$ intersections with α and replace E by the disjoint disc $E_1 \cup D_1$ which intersects α in at most $j/2 + j'/2 < j'$ points.

This shows the above claim and completes the proof of the lemma. Note that as before, if D, E are disjoint from a simple closed multicurve β then the construction yields discs which are disjoint from β and shows that $d_{\mathcal{S},\beta}(D, E) \leq 2k + 4$. $\square$

Remark: The arguments in this section use in the fact that any simple surgery of a disc at an outer component of another disc yields an essential disc. They are not valid for handlebodies with marked points on the boundary.

3. Distance in the curve graph

The purpose of this section is to establish some estimate for the distance in the curve graph which will be essential for a geometric description of the superconducting disc graph.

We fomulate all results in this section for an arbitrary orientable surface S of genus $g \geq 0$ with $n \geq 0$ punctures and $3g - 3 + n \geq 2$. The idea is to use *train tracks* on S. We refer to [PH92] for all the basic notions and constructions regarding train tracks.

A train track η (which may just be a simple closed curve) is *carried* by a train track τ if there is a map $F : S \to S$ of class C^1 which is isotopic to the identity, with $F(\eta) \subset \tau$ and such that the restriction of the differential dF of F to the tangent line of η vanishes nowhere. Write $\eta \prec \tau$ if η is carried by τ. If $\eta \prec \tau$ then the image of η under a carrying map is a *subtrack* of τ which does not depend on the choice of the carrying map. Such a subtrack is a subgraph of τ which is itself a train track.

A train track τ is called *large* [MM99] if each complementary component of τ is either simply connected or a once punctured disc. A simple closed curve η carried by τ *fills* τ if the image of η under a carrying map is all of τ. A *diagonal extension* of a large train track τ is a train track η which can be obtained from τ by subdividing some complementary components which are not trigons or once punctured monogons.

A *trainpath* on τ is an immersion $\rho : [k, \ell] \to \tau$ which maps every interval $[m, m + 1]$ diffeomorphically onto a branch of τ. We say that ρ is *periodic* if $\rho(k) = \rho(\ell)$ and if the inward pointing tangent of ρ at $\rho(k)$ equals the outward pointing tangent of ρ at $\rho(\ell)$. Any simple closed curve carried by a train track τ defines a periodic trainpath and a *transverse measure* on τ. The space of transverse measures on τ is a cone in a finite dimensional real vector space. Each of its extreme rays is spanned by a *vertex cycle* which is a simple closed curve carried by τ. A vertex cycle defines a periodic trainpath which passes through every branch at most twice, in opposite direction [H06, Mo03].

Lemma 3.1. *Let $(\tau_i)_{i=0}^n$ be a sequence of train tracks such that for every $i < n$, $\tau_i \prec \tau_{i+1}$ and that moreover every vertex cycle of τ_i fills a large subtrack of τ_{i+1}. Let α be a simple closed curve carried by τ_0. Then the total weight put on τ_n by α is not smaller than 2^n.*

Proof. Let τ be any train track and let α be a simple closed curve carried by τ. Then α defines an integral transverse measure ν on τ (i.e. the counting measure couting the number of preimages of an edge under a carrying map). We claim that $\nu = \sum_j a_j \sigma_j$ where σ_j is a vertex cycle of τ and $a_j \geq 1$ is an integer for all j.

To see that this is indeed the case orient α in an arbitrary way. Then the oriented curve α defines a periodic trainpath ρ on τ. Let $k < \ell$ be integers so that $\rho(k) = \rho(\ell)$ and that $\rho[k, \ell]$ is a periodic trainpath which does not contain any periodic subpath. Then $\rho[k, \ell]$ is a vertex cycle for τ (see [MM99]), and $\nu - \rho[k, \ell]$ is an integral transverse measure on τ. Such a measure defines a weighted simple closed multicurve α' carried by τ. Apply the above discussion to α'. After finitely many steps we obtain a decomposition of the transverse weight of α as claimed.

Let $(\tau_i)_{i=0}^n$ be a sequence so that $\tau_i \prec \tau_{i+1}$ and that for each i, every vertex cycle of τ_i fills a large subtrack of τ_{i+1}. Let α be any vertex cycle of τ_0. Then α defines a positive integral transverse measure ν on τ_1. Since $3g - 3 + m \geq 2$ and since α

fills a large subtrack of τ_1 by assumption, α is not a vertex cycle of τ_1. Thus by the discussion in the previous paragraph, its weight function can be represented in the form $\sum a_j \sigma_j$ where for each j, σ_j is a vertex cycle of τ_1 and where a_j is a positive integer with $\sum_j a_j \geq 2$.

By induction, we conclude that the weight function on τ_n defined by α can be decomposed as $\sum_j b_j \xi_j$ with vertex cycles ξ_j and $\sum_j b_j \geq 2^n$. The lemma follows. $\square$

Call a pair $\eta \prec \tau$ of large train tracks *wide* if the following holds true.
  (1) A vertex cycle of η fills a large subtrack of τ.
  (2) A simple closed curve carried by a diagonal extension of a large subtrack of η intersects every vertex cycle of τ.

We have

Lemma 3.2. *If $\sigma \prec \eta \prec \tau$ and if the pair $\eta \prec \tau$ is wide then $\sigma \prec \tau$ is wide.*

Proof. Let $\sigma \prec \eta \prec \tau$ be as in the lemma. Let α be a vertex cycle on σ. Then α defines a transverse measure on η which can be represented in the form $\sum_i a_i \beta_i$ where $a_i > 0$ and β_i is a vertex cycle on η. The image of α under a carrying map $\sigma \to \tau$ is the union of the images of the vertex cycles β_i under a carrying map $\eta \to \tau$. By assumption, each β_i fills a large subtrack of τ and hence α fills a large subtrack of τ as well.

Now let σ' be a large subtrack of σ and let ξ be a diagonal extension of σ'. Then the carrying map $\sigma \to \eta$ maps σ' onto a large subtrack η' of η, and it maps ξ to a diagonal extension ζ of η'. A simple closed curve α carried by ξ is carried by ζ. In particular, since $\eta \prec \tau$ is wide, α intersects every vertex cycle of τ. From this the lemma follows. $\square$

Recall from the introduction the definition of the curve graph $\mathcal{CG}$ of the surface S. For the formulation of the following observation, note as in the proof of Lemma 3.2 that if $\eta \prec \tau$ and if η' is a large subtrack of η then a diagonal extension of η' is carried by a diagonal extension of a large subtrack of τ.

Lemma 3.3. *Let $\tau_0 \prec \tau_1 \prec \tau_2$ be a triple of large train tracks such that the pairs $\tau_0 \prec \tau_1$ and $\tau_1 \prec \tau_2$ are wide. Then the distance in the curve graph between a simple closed curve carried by τ_0 and a vertex cycle of τ_2 is at least three.*

Proof. Let $\tau_0 \prec \tau_1 \prec \tau_2$ be as in the lemma. Let α be a simple closed curve carried by τ_0. Since the pair $\tau_0 \prec \tau_1$ is wide, α fills a large subtrack τ_1' of τ_1. Lemma 4.4 of [MM99] shows that a simple closed curve β which is disjoint from α is carried by a diagonal extension of τ_1'. Since the pair $\tau_1 \prec \tau_2$ is wide, β intersects every vertex cycle of τ_2. In other words, the distance in $\mathcal{CG}$ between β and a vertex cycle of τ_2 is at least two. The lemma follows. $\square$

For every large train track τ there is a *dual bigon track* τ^*. This bigon track is large, and a simple closed curve ζ carried by τ^* *hits τ efficiently* [PH92]. In particular, ζ defines (non-uniquely) a *tangential measure* ν on τ which is a weight function on τ. Let ι be the intersection form on measured geodesic laminations. If μ is the transverse measure on τ defined by a simple closed curve ξ carried by τ then

$$(2) \qquad\qquad\qquad \iota(\xi, \zeta) = \sum_b \mu(b) \nu(b)$$

where the sum is over all branches of τ. Moreover, if $\eta \prec \tau$ then $\tau^* \prec \eta^*$.

A *splitting and shifting sequence* is a sequence (τ_i) of train tracks such that for each i, τ_i can be obtained from τ_{i+1} by a sequence of *shifts* followed by a single *split*. (Note that we reverse indices here in contrast to the customary notations.) If $\eta \prec \tau$ then τ can be connected to η by a splitting and shifting sequence [PH92].

Lemma 3.3 together with the main result of [H11] are used to show

Proposition 3.4. *There is a number $\theta > 0$ with the following property. Let $(\tau_i)_{i=0}^n$ be a splitting and shifting sequence of large train tracks. Define inductively a sequence $j_0 = 0 < \cdots < j_k = n$ as follows. If j_i has already been determined, let $j_{i+1} \in (j_i, n]$ be the largest number such that the pair $\tau_{j_i} \prec \tau_{j_{i+1}-1}$ is not wide. Then the distance in the curve graph between a vertex cycle of τ_n and a simple closed curve carried by τ_0 is at least $k/\theta - \theta$.*

Proof. Let $(\tau_i)_{i=0}^n$ be a splitting and shifting sequence of large train tracks as in the lemma. Let ω be a simple closed curve carried by τ_0 and let ζ be a simple closed curve which hits τ_n efficiently. We may assume that ω, ζ jointly fill up S. Thus if we replace ζ by the weighted simple closed curve $\zeta' = \zeta/\iota(\omega, \zeta)$ then for all t the pair $(e^{-t}\omega, e^t\zeta')$ defines an *area one quadratic differential* q_t, and $t \to q_t$ is the unit cotangent line of a Teichmüller geodesic.

For a number $\delta > 0$, a simple closed curve α on S is called δ-*wide* for an area one quadratic differential q if α is the core curve of an annulus whose width with respect to the singular euclidean metric defined by q is at least δ (see [H11] for a discussion).

For each $k \leq n$ let $t_k \in \mathbb{R}$ be such that $\omega(\tau_k) = e^{t_k}$ (here $\omega(\tau_k)$ is the total weight which ω puts on τ_k). The transverse measure on τ_k of total weight one defined by $e^{-t_k}\omega$ can be represented in the form

$$e^{-t_k}\omega = \sum_j a_j^k \beta_j^k$$

where β_j^k are distinct vertex cycles of τ_k and where $a_j^k > 0$.

Now the total weight that a vertex cycle puts on a large train track τ is uniformly bounded, and the number of vertex cycles of a large train track is uniformly bounded as well. Therefore there is a number $\delta > 0$ only depending on the topological type of S and there is a number j so that $a_j^k \geq \delta$. Write $\beta^k = \beta_j^k$. Then the q_{t_k}-length of every simple closed curve which intersects β^k is at least δ (see [H06] for a discussion). In other words, β^k is δ-wide for q_{t_k}.

Since τ_k can be obtained from τ_{k+1} by a sequence of shifts followed by a single split, the number $s = |t_{k+1} - t_k|$ is uniformly bounded. Hence there is a number $\delta' > 0$ only depending on the topological type of S such that for every k and every $t \in [t_k, t_{k+1}]$, a δ-wide curve β for q_{t_k} is δ'-wide for q_t. This means that if we associate to $t \in [t_k, t_{k+1})$ the curve β^k then the thus defined map associates to each $t \in [t_0, t_n]$ a δ'-wide curve for q_t.

Let $(j_i) \subset [0, n]$ be a sequence as in the proposition. By construction and Lemma 3.2, for all i and every $p \leq j_i$, $q \geq j_{i+1}$ the pair $\tau_p \prec \tau_q$ is wide. Lemma 3.3 shows that for every i and every $t \leq t_{j_i}$, every $s \geq t_{j_{i+2}}$ the distance in the curve graph between β_t and β_s is at least three. The proposition now follows from Lemma 3.1 and the main result of [H11]. $\qquad\square$

4. QUASI-GEODESICS IN THE SUPERCONDUCTING DISC GRAPH

In this section we complete the proof of Theorem 2 from the introduction. It is based on the results in Section 2 and Section 3 and a construction from [MM04].

Let again H be a handlebody of genus $g \geq 2$. Let $D, E \subset H$ be two discs in normal position. Let E' be an outer component of $E - D$ and let D_1 be a disc obtained from D by simple surgery at E'. If we denote by α the intersection of $\partial E'$ with ∂H then up to isotopy, the boundary ∂D_1 of the disc D_1 contains α as an embedded subarc. Moreover, α is disjoint from E. In particular, given an outer component E'' of $E - D_1$, there is a distinguished choice for a disc D_2 obtained from D_1 by simple surgery at E''. The disc D_2 is determined by the requirement that α is *not* a subarc of ∂D_2. For an outer component of $E - D_2$ there is a distinguished choice for a disc D_3 obtained from D_2 by simple surgery at an outer component of $E - D_2$ etc. We call a surgery sequence of this form a *nested* surgery path in direction of E.

The following result is due to Masur and Minsky (this is Lemma 4.2 of [MM04] which is based on Lemma 4.1 and the proof of Theorem 1.2 in that paper).

Proposition 4.1. *Let $D, E \subset \partial H$ be any discs in normal position and let $D = D_0, \dots, D_m$ be a nested surgery path in direction of E. Then for each $i \leq m$ there is a train track τ_i on ∂H such that the following holds true.*

(1) *τ_i carries ∂E and ∂E fills up τ_i.*
(2) *$\tau_i \prec \tau_{i+1}$.*
(3) *The disc D_i intersects τ_i in at most two points.*
(4) *If there is a simple closed multicurve β disjoint from $E \cup D$ then τ_i is disjoint from β for all i.*

We use Proposition 3.4 and Proposition 4.1 to complete the proof of Theorem 2 from the introduction (compare [MS10]).

Theorem 4.2. *The vertex inclusion defines a quasi-isometric embedding $\mathcal{SDG} \to \mathcal{CG}$.*

Proof. As before, let $d_{\mathcal{S}}$ be the distance in $\mathcal{SDG}$. Let D, E be two discs. By Proposition 4.1 there is a nested surgery path $D = D_0, \dots, D_m$ connecting the disc $D_0 = D$ to a disc D_n which is disjoint from E, and there is a sequence $(\tau_i)_{i=0}^n$ of train tracks on ∂H such that $\tau_i \prec \tau_{i+1}$ for all $i < n$ and that D_i intersects τ_i in at most two points.

By Proposition 3.4 it suffices to show the existence of a number $b > 0$ with the following property. Let $i < k$ be such that the pair $\tau_i \prec \tau_k$ is *not* wide; then $d_{\mathcal{S}}(D_i, D_k) \leq b$.

By the results of [PH92], there is a splitting and shifting sequence $\tau_i = \eta_0 \prec \cdots \prec \eta_s = \tau_k$ connecting τ_i to τ_k and a sequence $0 = u_i < \cdots < u_k = s$ so that $\eta_{u_q} = \tau_q$ for $i \leq q \leq k$. We distinguish two cases.

Case 1: There is a vertex cycle α of τ_i which does not fill a large subtrack of τ_k.

For each $p \leq s$ let $\zeta_p \prec \eta_p$ be the subtrack of η_p filled by α. Then ζ_p is not large. The union Y_p of ζ_p with the simply connected components of $\partial H - \zeta_p$ and the once punctured disc components of $\partial H - \zeta_p$ is a proper subsurface of ∂H for all p. The boundary of Y_p can be realized as a union of simple closed curves which are embedded in η_p (but with cusps). The carrying map $\eta_{p-1} \to \eta_p$ maps Y_{p-1} into Y_p. In particular, either the boundary of Y_{p-1} coincides with the boundary of Y_p

or Y_{p-1} is a *proper* subsurface of Y_p. In other words, the subsurfaces Y_p are nested, and hence their number is bounded from above by a universal constant $h > 0$.

Since ∂D_q intersects τ_q in at most two points, the number of intersections between ∂D_q and ∂Y_{u_q} is bounded from above by a universal constant $\chi > 0$. As a consequence, there are h simple closed curves $c_1, \ldots, c_h$ on ∂H so that for every $q \in [i, k]$ there is some $r(q) \in \{1, \ldots, h\}$ with

$$\iota(\partial D_q, c_{r(q)}) \leq \chi.$$

Each of the curves c_j is a fixed boundary component of one of the subsurfaces Y_q.

By reordering, assume that $r(i) = 1$. Let v_1 be the maximum of all numbers $q \in [i, k]$ such that $r(v_1) = 1$. Proposition 2.4 shows that $d_{\mathcal{S}}(D_i, D_{v_1}) \leq 2\chi + 4$. On the other hand, we have $d_{\mathcal{S}}(D_{v_1}, D_{v_1+1}) = 1$. Again by reordering, assume that $r(v_1 + 1) = 2$ and repeat this construction with the discs $D_{v_1+1}, \ldots, D_k$ and the curves $c_2, \ldots, c_h$. In $a \leq h$ steps we construct in this way an increasing sequence $i \leq v_1 < \cdots < v_a = k$ such that $d_{\mathcal{S}}(D_{v_u}, D_{v_{u+1}}) \leq 2\chi + 5$ for all $u \leq a$. This implies that

$$d_{\mathcal{S}}(D_i, D_k) \leq h(2\chi + 5)$$

which is what we wanted to show.

Case 2: There is a large subtrack τ_i' of τ_i, a diagonal extension ζ_i of τ_i' and a simple closed curve α carried by ζ_i which is disjoint from a vertex cycle of τ_k.

Since ζ_i is a diagonal extension of a large subtrack of τ_i and since D_i intersects τ_i in at most two points, the intersection number between ∂D_i and ζ_i is uniformly bounded.

For each $j \in [i, k]$, the image of τ_i' under a carrying map $\tau_i \to \tau_j$ is a large subtrack τ_j' of τ_j, and there is a diagonal extension ζ_j of τ_j' which carries α. We may assume that $\zeta_u \prec \zeta_j$ for $u \leq j$. Note that for all $j \in [i, k]$ the disc D_j intersects ζ_j in a uniformly bounded number of points.

If the subtrack of ζ_k filled by α is not large then the argument used in Case 1 above can be applied to a splitting and shifting sequence connecting ζ_i to ζ_k which passes through the diagonal extensions ζ_j of the large subtracks τ_j' of τ_j $(i \leq j \leq k)$. It shows that $d_{\mathcal{S}}(D_i, D_k)$ is uniformly bounded.

We are left with the case that the simple closed curve α fills a large subtrack ζ_k' of ζ_k but is disjoint from a vertex cycle β of τ_k. Using the notations from the previous two paragraphs, let $j \geq i$ be the largest number so that α does *not* fill a large subtrack of ζ_{j-1}. The discussion in the previous paragraph shows that $d_{\mathcal{S}}(D_i, D_{j-1})$ is uniformly bounded. Since $d_{\mathcal{S}}(D_{j-1}, D_j) = 1$ we are left with showing that $d_{\mathcal{S}}(D_j, D_k)$ is uniformly bounded.

For $j \leq p \leq k$ the curve α fills a large subtrack ζ_p' of ζ_p. Note that we have $\zeta_p' \prec \zeta_k'$ for all p. Since α and β are disjoint, Lemma 4.4 of [MM99] shows that there is a diagonal extension $\hat{\zeta}_p$ of ζ_p' which carries β. Since $\zeta_p' \prec \zeta_k'$ we may assume that $\hat{\zeta}_p \prec \hat{\zeta}_k$ for $j \leq p \leq k$. However, β is a vertex cycle of τ_k and therefore the total weight which β puts on $\hat{\zeta}_k$ is uniformly bounded. Since $\hat{\zeta}_p$ carries β and since $\hat{\zeta}_p \prec \hat{\zeta}_k$, the total weight which β puts on $\hat{\zeta}_p$ is uniformly bounded as well.

Now for each p the disc D_p intersects $\hat{\zeta}_p$ in a uniformly bounded number of points. As a consequence, β intersects each of the discs D_p $(j \leq p \leq k)$ in a uniformly bounded number of points. From Proposition 2.4 we deduce that $d_{\mathcal{S}}(D_j, D_k)$ is uniformly bounded. This completes the proof of the theorem. $\square$

For a simple closed multicurve β on ∂H which is not discbusting and does not contain a discbounding component define the *superconducting disc graph* $\mathcal{SDG}(\beta)$ of $\partial H - \beta$ to be the graph whose vertices are discs D with boundary $\partial D \subset \partial H - \beta$. Two such discs D, E are connected by an edge of length one if one of the following three possibilities is satisfied.

(1) D, E are disjoint.
(2) There is an essential simple closed curve α in $\partial H - \beta$ (i.e. an essential curve not homotopic to a component of β) which is disjoint from $\partial D \cup \partial E$.
(3) There is a discbusting I-bundle $\gamma \subset \partial H - \beta$ which intersects each of the discs D, E in precisely two points.

Note that $\mathcal{ECG}(\beta)$ is connected. If $\partial H - \beta$ decomposes into at least two connected components which contain boundaries of discs then the diameter of $\mathcal{EDG}(\beta)$ equals two.

For an essential connected subsurface X of ∂H denote by $\mathcal{CG}(X)$ the curve graph of X. The proof of Theorem 4.2 also yields

Corollary 4.3. *There is a number $L_0 > 1$ with the following property. Let $\beta \subset \partial H$ be a multicurve which is not discbusting and does not contain a discbounding component. If there is a unique component X of $\partial H - \beta$ which contains the boundary of some disc in H then the vertex inclusion defines an L_0-quasi-isometric embedding $\mathcal{SDG}(\beta) \to \mathcal{CG}(X)$.*

Note that in the statement of the corollary, the diameter of $\mathcal{SDG}(\beta)$ may be finite.

A hyperbolic geodesic metric space X admits a *Gromov boundary*. This boundary is a topological space on which the isometry group of X acts as a group of homeomorphisms.

Let $\mathcal{L}$ be the space of all *geodesic laminations* on ∂H equipped with the *coarse Hausdorff topology*. In this topology, a sequence (μ_i) converges to a lamination μ if every accumulation point of (μ_i) in the usual Hausdorff topology contains μ as a sublamination. Note that the coarse Hausdorff topology on $\mathcal{L}$ is not T_0, but its restriction to the subspace $\partial \mathcal{CG} \subset \mathcal{L}$ of all minimal geodesic laminations which fill up ∂H is Hausdorff. The space $\partial \mathcal{CG}$ equipped with the coarse Hausdorff topology can naturally be identified with the Gromov boundary of the curve graph [H06].

Let $\partial \mathcal{SDG} \subset \partial \mathcal{CG}$ be the closed subset of all geodesic laminations which are limits in the coarse Hausdorff topology of boundaries of discs in H. The handlebody group $\mathrm{Map}(H)$ acts on $\partial \mathcal{CG}$ as a group of transformations preserving the subset $\partial \mathcal{SGD}$. The Gromov boundary of $\mathcal{SDG}$ can now easily be determined from Theorem 4.2.

Namely, since the vertex inclusion $\mathcal{SDG} \to \mathcal{CG}$ defines a quasi-isometric embedding, the Gromov boundary of $\mathcal{SDG}$ is the subset of the Gromov boundary of $\mathcal{CG}$ of all endpoints of quasi-geodesic rays in $\mathcal{CG}$ which are contained in $\mathcal{SDG}$. By the main result of [H06], a simplicial quasi-geodesic ray $\gamma : [0, \infty) \to \mathcal{CG}$ defines the endpoint lamination $\nu \in \partial \mathcal{CG}$ if and only if the curves $\gamma(i)$ converge as $i \to \infty$ in the coarse Hausdorff topology to ν. As a consequence, the Gromov boundary of $\mathcal{SDG}$ is a subset of $\partial \mathcal{SDG}$.

We claim that this is a closed subset of $\partial \mathcal{CG}$. Namely, let (ν_i) be a sequence in this set which converges in $\partial \mathcal{CG}$ to a lamination ν. Let ∂D be the boundary of a disc and let $\gamma : [0, \infty) \to \mathcal{CG}$ be a geodesic ray issuing from $\gamma(0) = \partial D$ with endpoint ν. By hyperbolicity of $\mathcal{CG}$, there is a number $L > 0$ and for every $k \geq 0$ there is

some $i(k) > 0$ such that a geodesic in $\mathcal{SDG}$ connecting $\gamma(0)$ to $\nu_{i(k)}$ passes through the L-neighborhood of $\gamma(k)$. Since $k > 0$ was arbitrary, this implies that the entire geodesic ray γ is contained in the L-neighborhood of the subset $\mathcal{SDG}$ of $\mathcal{CG}$. Using once more hyperbolicity, this shows that there is a uniform quasi-geodesic in $\mathcal{CG}$ connecting $\gamma(0)$ to ν which is entirely contained in $\mathcal{SDG}$. But this just means that ν is contained in the Gromov boundary of $\mathcal{SDG}$.

Now the handlebody group $\mathrm{Map}(H)$ acts on both $\mathcal{SDG}$ and $\mathcal{CG}$ as a group of isometries. Therefore the Gromov boundary of $\mathcal{SDG}$ is a closed $\mathrm{Map}(H)$-invariant subset of $\partial \mathcal{SDG}$. Together with the following observation (which is essentially contained in Theorem 1.2 of [M86]), we conclude that $\partial \mathcal{SDG}$ is indeed the Gromov boundary of $\mathcal{SDG}$.

Lemma 4.4. *The action of the handlebody group* $\mathrm{Map}(H)$ *on* $\partial \mathcal{SDG}$ *is minimal.*

Proof. Let (∂D_i) be a sequence of boundaries of discs converging in the coarse Hausdorff topology to a geodesic lamination $\mu \in \partial \mathcal{SDG}$. For each i let E_i be a disc which is disjoint from D_i. Since the space of geodesic laminations equipped with the usual Hausdorff topology is compact, up to passing to a subsequence the sequence (∂E_i) converges in the Hausdorff topology to a geodesic lamination ν which does not intersect μ. Now μ is minimal and fills up ∂H and therefore the lamination ν contains μ as a sublamination. This just means that (∂E_i) converges in the coarse Hausdorff topology to μ.

Every separating disc is disjoint from some non-separating disc. Thus the above argument shows that every $\mu \in \partial \mathcal{SDG}$ is a limit in the coarse Hausdorff topology of a sequence of non-separating discs. However, the handlebody group acts transitively on non-separating discs. Minimality of the action of $\mathrm{Map}(H)$ on $\partial \mathcal{SDG}$ follows. $\qquad\square$

Thus we have shown

Corollary 4.5. $\partial \mathcal{SDG}$ *is the Gromov boundary of* $\mathcal{SDG}$.

5. Hyperbolic extensions of hyperbolic graphs

In this section we give a condition for an extension of a hyperbolic graph to be hyperbolic. This condition will repeatedly be used in the later sections. It is related to the work of Farb [F98].

To begin with, let $(\mathcal{G}, d)$ be any (not necessarily locally finite) metric graph (i.e. edges have length one). For a given family $\mathcal{H} = \{H_c \mid c \in \mathcal{C}\}$ of complete connected subgraphs of $\mathcal{G}$ define the $\mathcal{H}$-*electrification* of $\mathcal{G}$ to be the metric graph $(\mathcal{EG}, d_{\mathcal{E}})$ which is obtained from $\mathcal{G}$ by adding vertices and edges as follows. For each $c \in \mathcal{C}$ there is a unique vertex $v_c \in \mathcal{EG} - \mathcal{G}$. This vertex is connected with each of the vertices of H_c by a single edge of length one, and it is not connected with any other vertex. Note that an edge-path γ in $\mathcal{EG}$ which passes through $\gamma(k) = v_c$ for some $c \in \mathcal{C}$ satisfies $\gamma(k-1) \in H_c, \gamma(k+1) \in H_c$.

We say that the family $\mathcal{H}$ is r-*bounded* for a number $r > 0$ if $\mathrm{diam}(H_c \cap H_d) \leq r$ for $c \neq d \in \mathcal{C}$ where the diameter is taken with respect to the intrinsic path metric on H_c and H_d.

In the sequel all parametrized paths γ in $\mathcal{G}$ or $\mathcal{EG}$ are supposed to be *simplicial*. This means that the image of every integer is a vertex, and the image of an integral interval $[k, k+1]$ is an edge or constant. Call a quasi-geodesic γ in $\mathcal{EG}$ *efficient* if for every $c \in \mathcal{C}$ we have $\gamma(k) = v_c$ for at most one k.

Definition 5.1. The family $\mathcal{H}$ has the *bounded penetration property* if it is r-bounded for some $r > 0$ and if for every $L > 0$ there is a number $p(L) > 2r$ with the following property. Let γ be an efficient L-quasi-geodesic in $\mathcal{EG}$, let $c \in \mathcal{C}$ and let $k \in \mathbb{Z}$ be such that $\gamma(k) = v_c$. If the distance in H_c between $\gamma(k-1)$ and $\gamma(k+1)$ is at least $p(L)$ then every efficient L-quasi-geodesic γ' in $\mathcal{EG}$ with the same endpoints as γ passes through v_c. Moreover, if $k' \in \mathbb{Z}$ is such that $\gamma'(k') = v_c$ then the distance in H_c between $\gamma(k-1), \gamma'(k'-1)$ and between $\gamma(k+1), \gamma'(k'+1)$ is at most $p(L)$.

Let $\mathcal{H}$ be as in Definition 5.1. Define an *enlargement* $\hat{\gamma}$ of a simplicial L-quasi-geodesic $\gamma : [0, m] \to \mathcal{EG}$ as follows. Let $0 \leq k_1 < \cdots < k_s \leq m$ be those points such that $\gamma(k_i) = v_{c_i}$ for some $c_i \in \mathcal{C}$. Then $\gamma(k_i - 1), \gamma(k_i + 1) \in H_{c_i}$. Replace $\gamma[k_i - 1, k_i + 1]$ by a simplicial geodesic in H_{c_i} with the same endpoints.

The goal of this section is to show

Theorem 5.2. *Let $\mathcal{G}$ be a graph and let $\mathcal{H} = \{H_c \mid c\}$ be a bounded family of complete connected subgraphs of $\mathcal{H}$. Assume that the following conditions are satisfied.*

(1) *There is a number $\delta > 0$ such that each of the graphs H_c is δ-hyperbolic.*
(2) *The $\mathcal{H}$-electrification $\mathcal{EG}$ of $\mathcal{G}$ is hyperbolic.*
(3) *$\mathcal{H}$ has the bounded penetration property.*

Then $\mathcal{G}$ is hyperbolic. Enlargements of geodesics in $\mathcal{EG}$ are uniform quasi-geodesics in $\mathcal{G}$.

For the remainder of this section we assume that $\mathcal{G}$ is a graph with a family $\mathcal{H}$ of subgraphs which has the properties stated in Theorem 5.2.

For a number $R > 2r$ call $c \in \mathcal{C}$ R-*wide* for an efficient L-quasi-geodesic γ in $\mathcal{EG}$ if the following holds true. There is some $k \in \mathbb{Z}$ such that $\gamma(k) = v_c$, and the distance between $\gamma(k-1), \gamma(k+1)$ in H_c is at least R. Note that since $\mathcal{H}$ is r-bounded, c is uniquely determined by $\gamma(k-1), \gamma(k+1)$. If $R = p(L)$ as in Definition 5.1 then we simply say that c is *wide*.

Define the *Hausdorff distance* between two compact subsets A, B of a metric space to be the infimum of the numbers $r > 0$ such that A is contained in the r-neighborhood of B and B is contained in the r-neighborhood of A. We first observe

Lemma 5.3. *For every $L > 0$ there is a number $\kappa(L) > 0$ with the following property. Let γ_1, γ_2 be two efficient simplicial L-quasi-geodesics in $\mathcal{EG}$ connecting the same points in $\mathcal{G}$, with enlargements $\hat{\gamma}_1, \hat{\gamma}_2$. Then the Hausdorff distance in $\mathcal{G}$ between the images of $\hat{\gamma}_1, \hat{\gamma}_2$ is at most $\kappa(L)$.*

Proof. Let $\gamma : [0, n] \to \mathcal{EG}$ be an efficient simplicial L-quasi-geodesic with endpoints $\gamma(0), \gamma(n) \in \mathcal{G}$. Let $R > 2r$ be a fixed number. Assume that $c \in \mathcal{C}$ is not R-wide for γ. If there is some $u \in \{1, \ldots, n-1\}$ such that $\gamma(u) = v_c$ then $\gamma(u-1), \gamma(u+1) \in H_c$. Since c is not R-wide, $\gamma(u - 1)$ can be connected to $\gamma(u + 1)$ by an arc in H_c of length at most R. In particular, if no $c \in \mathcal{C}$ is R-wide for γ then an enlargement $\hat{\gamma}$ of γ is an $\hat{L}$-quasi-geodesic in $\mathcal{EG}$ for a universal constant $\hat{L} = \hat{L}(L, R) > 0$. Then $\hat{\gamma}$ is an $\hat{L}$-quasi-geodesic in $\mathcal{G}$ as well (note that the inclusion $\mathcal{G} \to \mathcal{EG}$ is 1-Lipschitz).

Let $\gamma_i : [0, m_i] \to \mathcal{EG}$ be efficient L-quasi-geodesics ($i = 1, 2$) with the same endpoints in $\mathcal{G}$. Assume that no $c \in \mathcal{C}$ is R-wide for γ_i ($i = 1, 2$). Let $\hat{\gamma}_i$ be an enlargement of γ_i. By the above discussion, the arcs $\hat{\gamma}_i$ are $\hat{L}(L, R)$-quasi-geodesics in $\mathcal{EG}$. In particular, by hyperbolicity of $\mathcal{EG}$, the Hausdorff distance in $\mathcal{EG}$ between

the images of $\hat{\gamma}_i$ is bounded from above by a universal constant $b-1 > 0$ (depending on L and R).

We have to show that the Hausdorff distance in $\mathcal{G}$ between these images is also uniformly bounded. For this let $x = \hat{\gamma}_1(u)$ be any vertex on $\hat{\gamma}_1$ and let $y = \hat{\gamma}_2(v)$ be a vertex on $\hat{\gamma}_2$ with $d_{\mathcal{E}}(x, y) \leq b$ (here as before, $d_{\mathcal{E}}$ is the distance in $\mathcal{EG}$). Let ζ be a geodesic in $\mathcal{EG}$ connecting x to y. We claim that there is a universal constant $\chi > 0$ such that no $c \in \mathcal{C}$ is χ-wide for ζ.

Namely, the concatenation $\xi = \gamma_2|[v, m] \circ \zeta \circ \gamma_1[0, u]$ (read from right to left) is an L'-quasi-geodesic in $\mathcal{EG}$ with the same endpoints as γ_1 where $L' > L$ only depends on L. Hence by the bounded penetration property, if $c \in \mathcal{C}$ is $(2p(L') + R)$-wide for ζ then c is R-wide for γ_1 which violates the assumption that no $c \in \mathcal{C}$ is wide for γ_1,

As a consequence of the above discussion, the length of an enlargement of ζ is bounded from above by a fixed multiple of $d_{\mathcal{E}}(\hat{\gamma}_1(u), \hat{\gamma}_1(v))$, i.e. it is uniformly bounded. As a consequence, if there are no R-wide curves for γ_i then indeed the Hausdorff distance in $\mathcal{G}$ between the images of the enlargements $\hat{\gamma}_1, \hat{\gamma}_2$ is bounded by a number only depending on L and R.

Now let $\gamma_j : [0, m_j] \to \mathcal{EG}$ be arbitrary efficient L-quasi-geodesics ($j = 1, 2$) with the same endpoints in $\mathcal{G}$. Then there are numbers $0 < u_1 < \cdots < u_k < m_1$ such that for every $i \leq k$, $\gamma_1(u_i) = v_{c_i}$ where $c_i \in \mathcal{C}$ is wide for γ_1, and there are no other wide points.

By the bounded penetration property, there is a sequence $0 < w_1 < \cdots < w_k < m_2$ such that $\gamma_2(w_i) = \gamma_1(u_i) = v_{c_i}$ for all i. Moreover, the distance in H_{c_i} between $\gamma_1(u_i - 1)$ and $\gamma_2(w_i - 1)$ and between $\gamma_1(u_i + 1)$ and $\gamma_2(w_i + 1)$ is at most $p(L)$.

For each $i \geq 0$, define a simplicial edge path $\zeta_i : [a_i, a_{i+1}] \to \mathcal{EG}$ connecting $\gamma_1(u_i + 1) \in H_{c_i}$ to $\gamma_1(u_{i+1} - 1) \in H_{c_{i+1}}$ as the concatenation of the following three arcs. A geodesic in H_{c_i} connecting $\gamma_1(u_i + 1)$ to $\gamma_2(w_i + 1)$ (whose length is at most $p(L)$), the arc $\gamma_2[w_i + 1, w_{i+1} - 1]$ and a geodesic in $H_{c_{i+1}}$ connecting $\gamma_2(w_{i+1} - 1)$ to $\gamma_2(u_{i+1} - 1)$. Let moreover $\eta_i = \gamma_1|[u_i + 1, u_{i+1} - 1]$ ($i \geq 0$). Then η_i, ζ_i are efficient uniform quasi-geodesics in $\mathcal{EG}$ with the same endpoints and without $p(L)$-wide points. By the first part of this proof, the Hausdorff distance in $\mathcal{EDG}$ between any choice of their enlargements is uniformly bounded.

However, there is an enlargement $\hat{\gamma}_1$ of γ_1 which can be represented as

$$\hat{\gamma}_1 = \hat{\eta}_k \circ \sigma_k \circ \cdots \circ \sigma_1 \circ \hat{\eta}_0$$

where σ_i is a geodesic in H_{c_i} connecting $\gamma_1(u_i - 1)$ to $\gamma_1(u_i + 1)$ and where $\hat{\eta}_i$ is an enlargement of the quasi-geodesic η_i, and similarly for γ_2.

For each i the distance in H_{c_i} between $\gamma_1(u_i - 1)$ and $\gamma_2(w_i - 1)$ is at most $p(L)$, and the same holds true for the distance between $\gamma_1(u_i + 1)$ and $\gamma_2(w_i + 1)$. Since H_{c_i} is δ-hyperbolic for a universal constant $\delta > 0$, the Hausdorff distance in H_{c_i} between any two geodesics connecting $\gamma_1(u_i - 1)$ to $\gamma_1(u_i + 1)$ and connecting $\gamma_2(w_i - 1)$ to $\gamma_2(w_i + 1)$ is uniformly bounded. This implies the lemma. $\qquad \square$

Let for the moment X be an arbitrary geodesic metric space. Assume that for every pair of points $x, y \in X$ there is a fixed choice of a path $\rho_{x,y}$ connecting x to y. The *thin triangle property* for this family of paths states that there is a universal number $C > 0$ so that for any triple x, y, z of points in X, the image of $\rho_{x,y}$ is contained in the C-neighborhood of the union of the images of $\rho_{y,z}, \rho_{z,x}$.

For two vertices $x, y \in \mathcal{G}$ let $\rho_{x,y}$ be an enlargement of a geodesic in $\mathcal{EG}$ connecting x to y. We have

Proposition 5.4. *The thin triangle inequality property holds for the paths $\rho_{x,y}$.*

Proof. Let x_1, x_2, x_3 be three vertices in $\mathcal{G}$ and for $i = 1, 2, 3$ let $\gamma_i : [0, m_i] \to \mathcal{EG}$ be a geodesic connecting x_i to x_{i+1}. By hyperbolicity of $\mathcal{EG}$ there is a number $L > 0$ and a point $y \in \mathcal{G}$ with the following property. For $i = 1, 2, 3$ let β_i be a geodesic in $\mathcal{EG}$ connecting x_i to y. Then for all i, $\alpha_i = \beta_{i+1}^{-1} \circ \beta_i$ is an L-quasi-geodesic connecting x_i to x_{i+1}.

In particular, if $c \in \mathcal{C}$ is $5p(L)$-wide for γ_i then c is $3p(L)$-wide for α_i, and α_i passes through v_c. Then v_c is $3p(L)$-wide for either β_i or β_{i+1}. Assume that this holds true for β_{i+1}. Since β_{i+1} is a subarc of α_{i+1}, c is $3p(L)$-wide for α_{i+1} and hence c is $p(L)$-wide for γ_{i+1}.

We distinguish two cases.

Case 1: There is some $c \in \mathcal{C}$ which is wide for each of the geodesics γ_i.

Then there are unique numbers u_i $(i = 1, 2, 3)$ such that $\gamma_i(u_i) = v_c$. The concatenation of $\gamma_i[0, u_i]$ with the inverse of $\gamma_{i+1}[0, u_{i+1}]$ is a geodesic ζ in $\mathcal{EG}$ connecting x_i to x_{i+1}. Thus by the bounded penetration property, applied to the geodesics ζ and γ_i, the distance in H_c between $\gamma_i(u_i + 1)$ and $\gamma_{i+1}(u_{i+1} - 1)$ is uniformly bounded.

By hyperbolicity of H_c, this implies that there is a number $R > 0$ not depending on c and there is a vertex $z \in H_c$ such that for all i, any geodesic in H_c connecting $\gamma_i(u_i - 1)$ to $\gamma_i(u_i + 1)$ passes through the R-neighborhood of z. We may assume that z is distinct from each of the points $\gamma_i(u_i - 1), \gamma_i(u_i + 1)$.

For each i define two simplicial paths ζ_i, η_i in $\mathcal{EG}$ connecting x_i to z by

$$\zeta_i[0, u_i] = \gamma_i[0, u_i], \ \zeta_i(u_i + 1) = z \text{ and}$$

$$\eta_i(s) = \gamma_{i-1}(m_{i-1} - s) \text{ for } s \in [0, m_{i-1} - u_{i-1}], \eta_i(m_{i-1} - u_{i-1} + 1) = z.$$

By construction, the curves ζ_i, η_i are efficient uniform quasi-geodesics in $\mathcal{EG}$. Thus by Lemma 5.3, the Hausdorff distance in $\mathcal{G}$ between any of their enlargements $\hat{\zeta}_i, \hat{\eta}_i$ is uniformly bounded. On the other hand, it follows from the choice of z and hyperbolicity that the Hausdorff distance between an enlargement of γ_i and an enlargement of the (non-efficient) simplicial quasi-geodesic $\eta_{i+1}^{-1} \circ \zeta_i$ is uniformly bounded. This implies the thin triangle property in this case.

Case 2: There is no $c \in \mathcal{C}$ which is wide for each of the geodesics γ_i.

Then by the discussion in the beginning of this proof, if $c \in \mathcal{C}$ is $5p(L)$-wide for γ_1 then c is $p(L)$-wide for exactly one of the geodesics γ_2, γ_3.

Let $c \in \mathcal{C}$ be $5p(L)$-wide for γ_1 and $p(L)$-wide for γ_3. Then we have $v_c = \gamma_1(u)$ for exactly one u, and $v_c = \gamma_3(w)$ for exactly one w. Using the bounded penetration property as in Case 1 above, we conclude that the distance in H_c between $\gamma_1(u+1)$ and $\gamma_3(w-1)$ is uniformly bounded. It now follows from the above discussion that the Hausdorff distance in $\mathcal{G}$ between the enlargements of $\gamma_1[0, u+1]$ and $\gamma_3[w-1, m_3]$ is uniformly bounded. Moreover, if there is some $k < u$ such that $\gamma_1(k) = v_c$ where $c \in \mathcal{C}$ is $5p(L)$-wide for γ_1, then c is wide for γ_3.

Let $u_0 \geq 0$ be the maximum of all numbers with the property that $\gamma_1(u_0) = v_c$ where $c \in \mathcal{C}$ is $5p(L)$-wide for γ_1 and $p(L)$-wide for γ_3. If no such number exists then we put $u_0 = 0$. Similarly, let $t_0 \leq m_1$ be the minimum of all numbers so that $\gamma_1(t_0) = v_e$ where $e \in \mathcal{C}$ is $3p(1)$-wide for γ_1 and wide for γ_2. If no such number exists put $t_0 = m_1$. Then $\gamma_1[u_0 + 1, t_0 - 1]$ is a (perhaps constant)

simplicial geodesic without $5p(L)$-wide points. Moreover, the Hausdorff distance in $\mathcal{EG}$ between $\gamma_1[0, u_0 + 1]$ and a subarc of γ_3 and between $\gamma_1[t_0 - 1, m_1]$ and a subarc of γ_2 is uniformly bounded. By hyperbolicity of $\mathcal{EG}$, this implies that there is some $s \in [u_0 + 1, t_0 - 1]$ such that the distance in $\mathcal{EG}$ between $z = \gamma(s)$ and both geodesics γ_2, γ_3 is uniformly bounded.

Let η be a geodesic in $\mathcal{EG}$ connecting z to x_3. We may assume that η intersects γ_1 only in z. By the choice of z and hyperbolicity of $\mathcal{EG}$, the concatenations $\eta \circ (\gamma_1|[0, s])$ and $(\gamma_1|[s, m_1]) \circ \eta^{-1}$ are efficient b-quasi-geodesics in $\mathcal{EG}$ for a universal number $b > 1$. Together with Lemma 5.3, the thin triangle property follows. $\square$

Now we are ready to show

Corollary 5.5. $\mathcal{G}$ *is hyperbolic. Enlargements of geodesics in* $\mathcal{EG}$ *are uniform quasi-geodesics in* $\mathcal{G}$.

Proof. For any pair (x, y) of vertices in $\mathcal{G}$ let $\eta_{x,y}$ be a reparametrization on $[0, 1]$ of the path $\rho_{x,y}$. By Proposition 3.5 of [H07], it suffices to show that there is some $m > 0$ such that the paths $\eta_{x,y}$ have the following properties (where now $d_{\mathcal{G}}$ is the distance in $\mathcal{G}$).

(1) If $d_{\mathcal{G}}(x, y) \leq 1$ then the diameter of $\eta_{x,y}[0, 1]$ is at most m.
(2) For x, y and $0 \leq s < t < 1$ the Hausdorff distance between $\eta_{x,y}[s, t]$ and $\eta_{\eta_{x,y}(s), \eta_{x,y}(t)}[0, 1]$ is at most m.
(3) For all vertices x, y, z the set $\eta_{x,y}[0, 1]$ is contained in the m-neighborhood of $\eta_{x,y}[0, 1] \cup \eta_{y,z}[0, 1]$.

Properties 1) and 2) above are immediate from Lemma 5.3. The thin triangle property 3) follows from Proposition 5.4 and Lemma 5.3. $\square$

6. Decreasing conductivity I: The electrified disc graph

The main goal of this section is to show Theorem 3 and the second part of Theorem 5 from the introduction. For the remainder of this section, we denote by $d_{\mathcal{CG}}$ the distance in the curve graph $\mathcal{CG}$ of ∂H, by $d_{\mathcal{S}}$ the distance in the superconducting disc graph $\mathcal{SDG}$ of H and by $d_{\mathcal{E}}$ the distance in the electrified disc graph $\mathcal{EDG}$ of H.

For a discbusting I-bundle $c \subset \partial H$ let $\mathcal{E}(c) \subset \mathcal{EDG}$ be the complete subgraph of $\mathcal{EDG}$ whose vertices are discs intersecting c in precisely two points and let $\mathcal{E} = \{\mathcal{E}(c) \mid c\}$.

The following observation is immediate from the definitions.

Lemma 6.1. $\mathcal{SDG}$ *is* 2*-quasi-isometric to the* $\mathcal{E}$*-electrification of* $\mathcal{EDG}$.

For the proof of Theorem 3 it now suffices to show that the family $\mathcal{E}$ of connected subgraphs of $\mathcal{SDG}$ satisfies the assumptions in Theorem 5.2.

We begin with showing that the graphs $\mathcal{E}(c)$ are δ-hyperbolic for some fixed $\delta > 0$. To this end, for a compact surface F with connected boundary ∂F define the *arc and curve graph* $\mathcal{C}'(F)$ as follows. Vertices of $\mathcal{C}'(F)$ are essential arcs with both endpoints in ∂F or essential simple closed curves in F. Here in the case that F is orientable, an essential simple closed curve α in F is a curve which is not contractible and not homotopic into the boundary. If F is non-orientable we require in addition that α does not bound a Moebius band. Two such arcs or curves are connected by an edge of length one if and only if they can be realized disjointly.

Similarly, the curve graph of F is defined to be the graph whose vertices are simple closed curves in F and where two such curves are connected by an edge of length one if either they can be realized disjointly or if F is a one-holed torus and they intersect in a single point. We have

Lemma 6.2. *The vertex inclusion defines a quasi-isometry of the curve graph of F onto $\mathcal{C}'(F)$.*

Proof. The lemma is well known if F is not a one-holed torus since in this case curves in the curve graph are connected by an edge if they can be realized disjointly.

Now let F be a one-holed torus and let c, d be two simple closed curves on F which intersect in a single point. Cutting F open along c yields a three-holed sphere, and d defines an embedded arc connecting the two boundary components corresponding to c. Thus there is an embedded arc α with endpoints on ∂F which is disjoint from c and intersects d in a single point. Similarly, there is an embedded arc β with endpoints on ∂F which is disjoint from α and d. As a consequence, the distance between c, d in the arc and curve-graph of F is at most three and hence the vertex inclusion defines a 3-Lipschitz map from the curve graph of F into the arc and curve-graph $\mathcal{C}'(F)$. The one-neighborhood of its image is all of $\mathcal{C}'(F)$.

We are left with showing that for every edge path in $\mathcal{C}'(F)$ connecting two simple closed curves there is an arc in the curve graph whose length is bounded from above by a uniform multiple of the arc. For this simply note that for any arc α in F with endpoints in ∂F there is a simple closed curve α' in F which is disjoint from α. This curve is the core curve of the annulus $F - \alpha$. If β is an arc disjoint from β, then the corresponding simple closed curve β' intersects α' in a single point. This yields the lemma. $\square$

As a consequence of Lemma 6.2, of [MM99] and its extension to non-orientable surfaces as found in the appendix of [BF07] we have

Theorem 6.3. $\mathcal{C}'(F)$ *is hyperbolic.*

On the other hand we have

Lemma 6.4. *Let c be a discbusting I-bundle with base surface F. Then $\mathcal{E}(c)$ is 2-quasi-isometric to the arc and curve-graph $\mathcal{C}'(F)$ of F.*

Proof. Let F be the base surface of the discbusting I-bundle c and let $\Pi : H \to F$ be the bundle projection. Then Π induces a map $\hat{\Pi} : \mathcal{E}(c) \to \mathcal{C}'(F)$ by associating to a disc D which intersects c in precisely two points the arc $\hat{\Pi}(D)$ in F such that D is the I-bundle over $\hat{\Pi}(D)$.

Two discs $D, E \in \mathcal{E}(c)$ are connected by an edge of length one if and only they are disjoint or if there is an essential simple closed curve on ∂H which is disjoint from both.

If $D, E \in \mathcal{E}(c)$ are disjoint then $\hat{\Pi}(D), \hat{\Pi}(E)$ are disjoint arcs in F with endpoints on ∂F. In particular, the distance in $\mathcal{C}'(F)$ between $\hat{\Pi}(D), \hat{\Pi}(E)$ equals one.

If $D, E \in \mathcal{E}(c)$ are disjoint from a simple closed curve α and if α is disjoint from c then $\Pi(\alpha)$ is an essential simple closed curve on F (perhaps covered twice if $\Pi(\alpha)$ is orientation reversing) which is disjoint from $\hat{\Pi}(D), \hat{\Pi}(E)$. Thus as before, the distance in $\mathcal{C}'(F)$ between $\hat{\Pi}(D), \hat{\Pi}(E)$ is at most two. On the other hand, if α intersects c then the projection of α into F consists of a collection of essential arcs

disjoint from D, E and once again, the distance in $\mathcal{C}'(F)$ between $\hat{\Pi}(D), \hat{\Pi}(E)$ does not exceed 2. As a consequence, the map $\hat{\Pi} : \mathcal{E}(c) \to \mathcal{C}'(F)$ is 2-Lipschitz.

On the other hand, the I-bundle over an arc $\alpha \in \mathcal{C}'(F)$ is a disc in $\mathcal{E}(c)$. Moreover, if α, β are two essential simple closed cuves in F then there is an arc with endpoints on ∂F which is disjoint from both α, β. As a consequence, for any geodesic in $\mathcal{C}'(F)$ connecting $\hat{\Pi}(D)$ to $\hat{\Pi}(E)$ there is an arc in $\mathcal{E}(c)$ of at most double length connecting D to E. The lemma follows. $\qquad \square$

We also have

Lemma 6.5. *There is a number $p > 0$ such that if c, d are distinct discbusting I-bundles then $\operatorname{diam}(\mathcal{E}(c) \cap \mathcal{E}(d)) \le p$.*

Proof. Let $c, d \subset \partial H$ be discbusting I-bundles. Since a simple closed curve disjoint from c is not discbusting, the curves c, d intersect.

Let α be a component of $d - c$. Then a disc $D \in \mathcal{E}(c) \cap \mathcal{E}(d)$ intersects α in at most two points. But this just means that the distance between $\partial D - c$ and α in the arc and curve graph of the base surface F of c is uniformly bounded. By Lemma 6.4, this implies that the diameter in $\mathcal{E}(c)$ of $\mathcal{E}(c) \cap \mathcal{E}(d)$ is uniformly bounded. $\qquad \square$

We are left with the verification of the bounded penetration property. To this end we begin with recalling from [MM00] the definition of a *subsurface projection*. Namely, let $X \subset \partial H$ be an incompressible, open connected subsurface which is distinct from S, a three-holed sphere and an annulus. As before, the arc and curve complex $\mathcal{C}'(X)$ of X is the complex whose vertices are isotopy classes of arcs with endpoints on ∂X or essential simple closed curves in X, and two such vertices are connected by an edge of length one if they can be realized disjointly. There is a projection π_X of the curve graph $\mathcal{CG}$ into the space of subsets of $\mathcal{C}'(X)$ which associates to a simple closed curve the homotopy classes of its intersection components with X. For every simple closed curve c, the diameter of $\pi_X(c)$ in $\mathcal{C}'(X)$ is at most one. Moreover, if c can be realized disjointly from X then $\pi_X(c) = \emptyset$.

If $A \subset S$ is an essential annulus then there is an arc complex $\mathcal{C}'(A)$ for A. Its vertices are homotopy classes of arcs with fixed endpoints on the two distinct boundary components of A. There is a subsurface projection π_A of $\mathcal{CG}$ into the space of subsets of A which is defined as follows [MM00]. Fix a hyperbolic metric on ∂H. Let $\tilde{A}$ be the annular cover of ∂H to which A lifts homeomorphically. There is a natural compactification $\hat{A}$ of $\tilde{A}$ to a closed annulus, obtained from the compactification of the hyperbolic plane. A simple closed curve γ on ∂H with an essential intersection with A lifts to a collection of disjoint arcs in $\hat{A}$ connecting the two distinct boundary components, one for each essential intersection of γ with A. Then $\pi_A(\gamma)$ is the union of these arcs, and $\pi_A(\gamma) = \emptyset$ if γ does not intersect A.

In the sequel we call a subsurface X of S *proper* if X is non-peripheral, incompressible, open and connected and different from a three-holed sphere or S.

As before, call a path γ in a metric graph G *simplicial* if γ maps each interval $[k, k+1]$ (where $k \in \mathbb{Z}$) isometrically onto an edge of G. The following lemma is a version of Theorem 3.1 of [MM00].

Lemma 6.6. *For every number $L > 1$ there is a number $\xi(L) > 0$ with the following property. Let X be a proper subsurface of ∂H and let γ be a simplicial path in $\mathcal{CG}$ which is an L-quasi-geodesic. If $\pi_X(v) \neq 0$ for every vertex v on γ then*

$$\operatorname{diam} \pi_X(\gamma) < \xi(L).$$

Proof. By hyperbolicity, for every $L > 1$ there is a number $p(L) > 0$ so that for every L-quasi-geodesic γ in $\mathcal{CG}$ of finite length, the Hausdorff distance between the image of γ and the image of a geodesic γ' with the same endpoints does not exceed $p(L)$.

Now let $X \subset \partial H$ be a proper subsurface. By Theorem 3.1 of [MM00], there is a number $M > 0$ with the following property. If ζ is any simplicial geodesic in $\mathcal{CG}$ and if $\pi_X(\zeta(s)) \neq \emptyset$ for all $s \in \mathbb{Z}$ then

$$\operatorname{diam}(\pi_X(\zeta)) \leq M.$$

Let $L > 1$, let $\gamma : [0, k] \to \mathcal{CG}$ be a simplicial path which is an L-quasi-geodesic and assume that

$$\operatorname{diam}(\pi_X(\gamma(0) \cup \gamma(k))) \geq 2M + 4L(p(L) + 3).$$

Let γ' be a geodesic with the same endpoints as γ and let $A \subset \mathcal{CG}$ be the set of all simple closed curves in the complement of X. The diameter of A is at most two. Theorem 3.1 of [MM00] shows that there is some $u \in \mathbb{R}$ such that $\gamma'(u) \in A$. Then γ passes through the $p(L)$-neighborhood of A.

Let $s + 1 \leq t - 1$ be the smallest and the biggest number, respectively, whose image under γ is contained in the $p(L)$-neighborhood of A. Then $\gamma[0, s]$ (or $\gamma[t, k]$) is contained in the complement of the $p(L)$-neighborhood of A. A geodesic connecting $\gamma(0)$ to $\gamma(s)$ (or connecting $\gamma(t)$ to $\gamma(k)$) is contained in the $p(L)$-neighborhood of $\gamma[0, s]$ (or of $\gamma[t, k]$) and hence it does not pass through A. In particular,

$$\operatorname{diam}(\pi_X(\gamma(0) \cup \gamma(s))) \leq M \text{ and } \operatorname{diam}(\pi_X(\gamma(t) \cup \gamma(k))) \leq M.$$

As a consequence, we have

(3) $$\operatorname{diam}(\pi_X(\gamma(s) \cup \gamma(t))) \geq 4L(p(L) + 3).$$

Since $d_{\mathcal{CG}}(\gamma(s+1), A) \leq p(L), d_{\mathcal{CG}}(\gamma(t-1), A) \leq p(L)$ we have $d_{\mathcal{CG}}(\gamma(s), \gamma(t)) \leq 2p(L) + 4$. Now γ is a simplicial L-quasi-geodesic and hence the length of $\gamma[s, t]$ is at most $L(2p(L) + 4) + L$. For all ℓ the curves $\gamma(\ell), \gamma(\ell+1)$ are disjoint and hence if $\gamma(\ell), \gamma(\ell+1)$ both intersect X then the diameter of $\pi_X(\gamma(\ell) \cup \gamma(\ell+1))$ is at most one. As a consequence, if $\gamma(\ell)$ intersects X for all ℓ then

$$\operatorname{diam}(\pi_X(\gamma(s) \cup \gamma(t))) \leq 2L(p(L) + 3).$$

This contradicts inequality (3) and completes the proof of the lemma.   □

Now we are ready to show

Lemma 6.7. *The family $\mathcal{E}$ has the bounded penetration property.*

Proof. Let $\mathcal{SDG}_0$ be the $\mathcal{E}$-electrification of $\mathcal{EDG}$. Recall that $\mathcal{SDG}_0$ is quasi-isometric to $\mathcal{SDG}$.

For some $L > 0$ let γ be an efficient L-quasi-geodesic in $\mathcal{SDG}_0$. Let γ' be the simplicial arc in $\mathcal{CG}$ which is obtained as follows. If $\gamma(k) = v_c$ for some discbusting I-bundle c then $\gamma(k-1) \in \mathcal{E}(c), \gamma(k+1) \in \mathcal{E}(c)$. Replace $\gamma[k-1, k]$ and $\gamma[k, k+1]$ by a geodesic in $\mathcal{CG}$ connecting $\gamma(k-1)$ to c and connecting c to $\gamma(k)$. repsectively. Note that the length of this geodesic is at most 3. Then γ' is an L'-quasi-geodesic in $\mathcal{CG}$ for a number $L' > 0$, and for each discbusting I-bundle c, the arc γ' passes at most once through c. We call γ' a *modification* of γ.

Now by Lemma 6.6 and Lemma 6.4, if the distance in $\mathcal{E}(c)$ between $\gamma(k-1)$ and $\gamma(k+1)$ is at least $\xi(qL')$ then any other efficient L-quasi-geodesic in $\mathcal{SDG}_0$ passes

through v_c as well. Moreover, if k' is such that $\gamma'(k') = v_c$ then the distance in $\mathcal{E}(c)$ between $\gamma(k-1), \gamma'(k'-1)$ and $\gamma(k+1), \gamma'(k'+1)$ is at most $\xi(qL')$. This shows the lemma. $\qquad\square$

We summarize the discussion as follows.

Corollary 6.8. $\mathcal{EDG}$ *is hyperbolic. Enlargements of geodesics in $\mathcal{SDG}$ are uniform quasi-geodesics. There is a number $p_1 > 0$ with the following property. Let $X \subset \partial H$ be a subsurface such that $\partial H - X$ is not discbusting. Let $D, E \subset H$ be any discs. If* $\mathrm{diam}(\pi_X(\partial D \cap \partial E)) \geq p_1$ *then any geodesic in $\mathcal{EDG}$ passes through a curve with boundary in $\partial H - X$. The bounded penetration property holds true.*

The following consequence is immediate from Corollary 6.8. For its formulation, let as before $d_{\mathcal{E}}$ be the distance in $\mathcal{EDG}$. For a set X and a number $C > 0$ define $\mathrm{diam}(X)_C$ to be the diameter of X if this diameter is at least C and let $\mathrm{diam}(X)_C = 0$ otherwise. If c is a discbusting I-bundle then let π^c be the projection of the boundary of a disc into the arc and curve-graph of the base surface F of c. Moreover, $\asymp$ means equality up to a universal multiplicative constant.

Corollary 6.9. *There is a number $C > 0$ such that*

$$d_{\mathcal{E}}(D, E) \asymp d_{\mathcal{CG}}(\partial D, \partial E) + \sum_c \mathrm{diam}(\pi^c(D, E))_C$$

where c runs through all discbusting I-bundles on ∂H.

Let again $\mathcal{L}$ be the space of all geodesic laminations on ∂H equipped with the coarse Hausdorff topology. For a discbusting I-bundle c let $\partial \mathcal{E}(c) \subset \mathcal{L}$ be the subset of all geodesic laminations which fill up $\partial H - c$ and which are limits in the coarse Hausdorff topology of boundaries of discs contained in $\mathcal{E}(c)$. Define

$$\partial \mathcal{EDG} = \partial \mathcal{SDG} \cup \bigcup_c \partial \mathcal{E}(c) \subset \mathcal{L}$$

where the union is over all discbusting I-bundles $c \subset \partial H$. We have

Corollary 6.10. *The Gromov boundary of $\mathcal{EDG}$ can naturally be identified with $\partial \mathcal{EDG}$.*

Proof. Note first that the space $\partial \mathcal{EDG}$ as defined above is invariant under the natural action of the group $\mathrm{Map}(H)$. Moreover, it is Hausdorff. Namely, a point $\lambda \in \partial \mathcal{EDG}$ either is a minimal geodesic lamination which fills ∂H, or it is a geodesic lamination with at most two minimal components which fills $\partial H - c$ for some discbusting I-bundle c. If ν is another such lamination and if ν is supported in $\partial H - c$ then ν intersects μ since both μ, ν fill $\partial H - c$. Otherwise ν fills the complement of a discbusting I-bundle $c' \neq c$ and hence once again, ν intersects μ. By the definition of the coarse Hausdorff topology, this implies that there are neighborhoods U of μ, V of ν so that any two laminations $\mu' \in U, \nu' \in V$ intersect. In particular, the neighborhoods U, V are disjoint.

Let $\gamma : [0, \infty) \to \mathcal{EDG}$ be a simplicial quasi-geodesic. By Corollary 6.8, we may assume that for each i the arc $\gamma[0, i]$, viewed as an arc in $\mathcal{SDG}$, is a uniform *unparametrized* quasi-geodesic in $\mathcal{SDG}$. As a consequence, if $\gamma(i) \to \infty$ in $\mathcal{SDG}$ then $\gamma(i)$ converges in $\mathcal{SDG} \cup \partial \mathcal{SDG}$ to a boundary point $\mu \in \partial \mathcal{SDG}$. By Corollary 4.5 and its proof, this implies that the boundaries $\partial \gamma(i)$ of the discs $\gamma(i)$ converge in the coarse Hausdorff topology to μ.

Now if as $i \to \infty$ the sequence $\gamma(i)$ is bounded in $\mathcal{SDG}$ then there exists a discbusting I-bundle $c \subset \partial H$ such that the diameter of $\pi^c(\gamma[0,\infty))$ is infinite. However, by Lemma 6.4, in this case the projection $\pi^c(\gamma[0,\infty))$ of $\gamma[0,\infty)$ defines an unbounded quasi-geodesic in the arc and disc-graph of the base surface F of c. Hence the simple closed curves $\partial\gamma(i)$ converge in the coarse Hausdorff topology to a lamination $\nu \in \partial\mathcal{E}(c)$. As a consequence, the Gromov boundary Λ of $\mathcal{EDG}$ can be identified with a subset of $\partial\mathcal{EDG}$. The fact that the natural inclusion $\Lambda \to \partial\mathcal{EDG}$ is surjective follows from the discussion in the proof of Corollary 4.5.

We are left with showing that the Gromov topology on $\mathcal{EDG}$ coincides with the coarse Hausdorff topology. To show that the coarse Hausdorff topology is coarser than the Gromov topology, consider a sequence γ_i of quasi-geodesic rays in $\mathcal{EDG}$ whose endpoints converge to the endpoint of a quasi-geodesic ray γ. Then as $i \to \infty$, longer and longer subsegments of γ_i uniformly fellow-travel longer and longer subsegments of γ. However, this implies that the endpoints of γ_i converge in the coarse Hausdorff topology to the endpoint of γ. That the Gromov topology is coarser than the coarse Hausdorff topology follows in exactly the same way (see [H06] for details). $\qquad\square$

Now let X be a (not necessarily connected) proper subsurface in ∂H which contains the boundary of some disc and which is bounded by simple closed curves which are not discbounding. Define the *electrified disc graph* $\mathcal{EDG}(X)$ to be the graph whose vertices are discs with boundary in X and where two such discs D, E are connected by an edge of length one if and only if either they can be realized discjointly or if there is an essential simple closed curve in X which is disjoint from both D, E. Note that the diameter of $\mathcal{EDG}(X)$ is infinite only if there are two discs D, E whose boundaries fill X.

The above discussion is valid for $\mathcal{EDG}(X)$ and shows

Corollary 6.11. $\mathcal{EDG}(X)$ *is hyperbolic. Enlargements of geodesics in* $\mathcal{SDG}(X)$ *are uniform quasi-geodesics in* $\mathcal{EDG}(X)$.

7. DECREASING CONDUCTIVITY II: THE DISC GRAPH

In this section we give an alternative proof of the following recent result of Masur and Schleimer [MS10].

Theorem 7.1. *The disc graph is hyperbolic.*

The idea of the proof is to successively enlarge the electrified disc graph with the procedure from Section 5. For this we first investigate another graph $\mathcal{EDG}(2)$ which lies between $\mathcal{DG}$ and $\mathcal{EDG}$. Its vertices are isotopy classes of essential discs in H, and two such discs D, E are connected by an edge of length one if and only if either D, E are disjoint or $\partial D, \partial E$ are disjoint from a multicurve $\beta \subset \partial H$ consisting of two not freely homotopic components.

For each non-separating simple closed curve c which is neither discbounding nor discbusting define $\mathcal{H}(c) = \mathcal{EDG}(\partial H - c)$. Then $\mathcal{H}(c)$ is a connected complete subgraph of $\mathcal{EDG}$. We have

Lemma 7.2. *The subgraphs* $\mathcal{H}(c)$ *are δ-hyperbolic for a fixed number $\delta > 0$. Moreover, if $c \neq d$ then the diameter in $\mathcal{EDG}(2)$ of the intersection $\mathcal{H}(c) \cap \mathcal{H}(d)$ is at most one.*

Proof. If c, d are distinct simple closed curves which are not discbounding then every disc whose boundary is disjoint from both c, d is disjoint from a multicurve which consists of at least two connected components. Thus the diameter of $\mathcal{H}(c) \cap \mathcal{H}(d)$ is at most 1. Hyperbolicity of $\mathcal{H}(c)$ was shown in Corollary 6.11. $\qquad\square$

Now $\mathcal{EDG}$ is quasi-isometric to the $\mathcal{H}$-electrification of $\mathcal{EDG}$. Thus to show that $\mathcal{EDG}(2)$ is indeed hyperbolic we are left with verifying the bounded penetration property.

However, we know from the discussion in Section 6 that an efficient quasi-geodesic γ in $\mathcal{EDG}$ intersects some region $\mathcal{H}(c)$ in a set with big diameter if and only if there is a large subsurface projection of the endpoints into $\partial H - c$. Together with Proposition 5.2, we conclude

Corollary 7.3. *$\mathcal{EDG}(2)$ is hyperbolic. Enlargements of geodesics in $\mathcal{EDG}$ are uniform quasi-geodesics in $\mathcal{EDG}(2)$.*

A repeated application of the above construction shows hyperbolicity of the disc graph. Namely, for $h \leq 3g - 3$ let $\mathcal{EDG}(h)$ be the graph whose vertices are discs and where two such discs D, E are connected by an edge of length one if and only if either they can be realized disjointly or if there is a multicurve β on ∂H with h components which is disjoint from $D \cup E$. Note that $\mathcal{EDG}(3g - 3) = \mathcal{DG}$. The above discussion implies that $\mathcal{EDG}(h)$ is hyperbolic if this is true for $\mathcal{EGG}(h - 1)$.

To summarize, let D, E be any two discs. Our construction shows that we obtain a uniform quasi-geodesic in $\mathcal{DG}$ connecting D to E as follows. Let $\gamma = \gamma_1$ be a geodesic in $\mathcal{EDG}$ connecting D to E. If $\gamma(i), \gamma(i + 1)$ are disjoint from a common simple closed curve c which is not discbounding then replace $\gamma'[i, i + 1]$ by a geodesic γ_2^i in $\mathcal{EDG}(\partial H - c)$ with the same endpoints. The resulting curve is a uniform quasi-geodesic in $\mathcal{EDG}(2)$ and will be called an *enlargement*. For each j construct an enlargement γ_3^j of γ_2^j. Proceed inductively. After $3g - 3$ steps we obtain a quasi-geodesic in $\mathcal{DG}$. From this we immediately obtain a distance formula for the disc graph as follows.

For $k \geq 0$ call a subsurface Y of ∂H *k-thick* if the following holds true.

(1) $\partial H - Y$ does not contain the boundary of any disc.
(2) The boundaries of discs fill up Y.
(3) There is a multicurve β with k components such that $\partial H - \beta$ is the union of Y with a collection of pairs of pants.

Note that a zero-thick subsurface of ∂H equals ∂H. A 1-thick subsurface Y is the complement of a simple closed curve which is not discbounding and such that the second property above holds true. A subsurface is called *thick* if it is k-thick for some $k \geq 0$.

The following corollary is now immediate from our construction.

Corollary 7.4. *There is a number $C > 0$ such that*

$$d_{\mathcal{DG}}(D, E) \asymp \sum_Y \mathrm{diam}(\pi_Y(E \cup D))$$

where Y runs through all thick subsurfaces of ∂H and the diameter is taken in the electrified disc graph.

For a thick subsurface Y of ∂H let $\partial \mathcal{EDG}(Y)$ be the Gromov boundary of $\mathcal{EDG}(Y)$. Define

$$\partial \mathcal{DG} = \cup_Y \partial \mathcal{EDG}(Y)$$

where the union is over all thick subsurfaces of ∂H and where this union is equipped with the coarse Hausdorff topology. As before we have

Corollary 7.5. *$\partial \mathcal{DG}$ is the Gromov boundary of $\mathcal{DG}$.*

8. Some examples

In this section we give some additional information on the various graphs investigated in the previous sections.

Let again H be a handlebody of genus $g \geq 2$. Define $\mathcal{DG}_{ns}$ to be the graph whose vertices are non-separating discs in H and where two vertices are connected by an edge of length one if and only if they can be realized disjointly.

Corollary 8.1. *The inclusion $\mathcal{DG}_{ns} \to \mathcal{DG}$ is an isometric embedding. In particular, $\mathcal{DG}_{ns}$ is hyperbolic.*

Proof. Let D, E be two non-separating discs and let $\gamma : [0, m] \to \mathcal{DG}$ be a geodesic connecting D to E. Let $i < m/2$ and assume that the disc $\gamma(2i + 1)$ is separating. Since γ is a geodesic, the boundaries $\partial\gamma(2i)$ and $\partial\gamma(2i+2)$ of the discs $\gamma(2i), \gamma(2i+2)$ are not disjoint and hence they are contained in the same component of $H - \partial\gamma(2i+1)$. But then there is a non-separating disc $\gamma'(2i + 1)$ which is disjoint from both $\gamma(2i)$ and $\gamma(2i + 2)$. Replacing $\gamma(2i + 1)$ by $\gamma'(2i + 1)$ defines a geodesic γ' in $\mathcal{DG}$ connecting D to E with the property that $\gamma'(2i + 1)$ is non-separating for all i.

Repeat this construction with the arc $\gamma'[1, m - 1]$ to construct a geodesic in $\mathcal{DG}$ connecting D to E which is contained in $\mathcal{DG}_{ns}$. $\square$

The disc graph of a handlebody H_0 with a marked point p on the boundary is the graph whose vertices are essential discs with boundary in $\partial H_0 - p$. Two such vertices are connected by an edge of length one if and only if they can be realized disjointly. The next proposition shows that disc graphs of handlebodies with spots are in general not hyperbolic. The construction was also observed by Schleimer (personal communication).

Proposition 8.2. *The disc graph of a handlebody of genus $g = 2m \geq 2$ with one marked point on the boundary is not hyperbolic.*

Proof. Let H be a handlebody of geneus $2m \geq 2$. Then there is an oriented surface F of genus m with connected boundary ∂F, and there is a homeomorphism Λ of the I-bundle $F \times I$ onto H. Let $c = \Lambda(\partial F) \subset \partial H$.

The I-bundle over every arc in F with boundary on ∂F is a disc in H. The mapping class group $\mathrm{Mod}(F)$ of F extends to a group of isotopy classes of homeomorphisms of $F \times I$ and hence it defines a subgroup of the handlebody group $\mathrm{Map}(H)$ of H. Let $\varphi \in \mathrm{Mod}(F) < \mathrm{Map}(H)$ be a pseudo-Anosov element. We may assume that φ is realized by a diffeomorphism of ∂H which fixes the curve c pointwise (and which will again be denoted by φ). Let $D \subset H$ be a non-separating disc which is the I-bundle over an embedded arc α in F with endpoints on ∂F. By the results in Section 7 (compare also [MS10]), the assignment $k \to \varphi^k D$ defines a quasi-geodesic in the disc graph of H.

Now let H_0 be the handlebody obtained from H by marking a point $p \in \partial H$. Assume that the marked point is contained in the simple closed curve c and is disjoint from ∂D. The *point pushing map ψ_c* of the simple closed curve c is the element of the handlebody group $\mathrm{Map}(H_0)$ which can be realized by a diffeomorphism fixing a

small neighborhood of c in H pointwise and which pushes the marked point about the simple closed curve c.

In other words, consider the Birman exact sequence

$$0 \to \pi_1(\partial H) \to \mathrm{Map}(H_0) \to \mathrm{Map}(H) \to 0$$

(we refer to [HH11] for a discussion). Then ψ_c is the image of the curve $c \in \pi_1(\partial H)$ in this sequence. Since $\varphi \in \mathrm{Map}(H)$ can be realized by a diffeomorphism which fixes a neighborhood of the curve c pointwise, the mapping class φ has a natural lift φ_0 to $\mathrm{Map}(H_0)$ which commutes with the mapping class ψ_c. Moreover, the disc D can be viewed as a disc in H_0. With this convention, for all k, ℓ we have $\psi_c^\ell \varphi_0^k D = \varphi_0^k \psi_c^\ell D$.

By Corollary 8.1, the graph $\mathcal{DG}_{ns}$ of non-separating discs in H is isometrically embedded in $\mathcal{DG}$. If we denote by $\mathcal{DG}_{ns}^0$ the graph of non-separating discs in H_0 then there is a natural puncture closing map $\Pi : \mathcal{DG}_{ns}^0 \to \mathcal{DG}_{ns}$. It maps a non-separating disc in H_0 to its isotopy class in H. Since disjoint discs in H_0 are mapped to disjoint discs in H, the map Π is distance non-increasing.

$\mathrm{Map}(H_0)$ acts as a group of isometries on $\mathcal{DG}_{ns}^0$. Hence there is a number $b > 0$ such that for all k, ℓ the distance in $\mathcal{DG}_{ns}^0$ between D and $\varphi_0^k(D)$ is bounded from above by $b|k|$. Since $k \to \varphi^k D$ is a quasi-geodesic in $\mathcal{DG}_{ns}$ and Π is distance non-increasing, we conclude that there is a number $L > 0$ such that for each $\ell > 0$ the assignment $k \to \psi_c^\ell \varphi_0^k(D)$ is an L-quasi-geodesic in $\mathcal{DG}_{ns}^0$.

As a consequence, if we can show the existence of a number $\kappa > 0$ such that for all k, ℓ the distance between $\psi_c^\ell \varphi_0^k D$ and $\cup_k \varphi_0^k D$ is contained in the interval $[\ell/2 - \kappa, \ell + \kappa]$ then the disc graph $\mathcal{DG}_{ns}^0$ is not hyperbolic.

The discs $\psi_c^\ell D, \psi_c^{\ell+1} D$ can be realized disjointly, so the upper distance bound is immediate. To show the lower distance bound, note that c is discbusting and hence every disc in H_0 intersects c. As a consequence, every disc defines a collection of arcs crossing through a punctured annulus neighborhood of c. Following the construction in [MM99], choose a hyperbolic Riemannian metric on ∂H_0 of finite volume (with the marked point as a cusp) and let $\hat{S}$ be the cover of ∂H_0 whose fundamental group is the fundamental group of the punctured annulus. Then the boundary of every disc E defines a collection of pairwise disjoint arcs in $\hat{S}$ connecting the two distinguished boundary components. A disc disjoint from E defines a collection of disjoint arcs. As a consequence, a path in the disc graph of H_0 can only linearly reduce intersections in $\hat{S}$ whence the lower distance bound. This completes the proof of the proposition. $\qquad\square$

However, the following is an immediate consequence of [KLS09].

Lemma 8.3. *The graph of non-separating discs on a solid torus with two marked points on the boundary is hyperbolic.*

Proof. Since the solid torus with one marked point on the boundary contains a unique non-separating disc, it follows from [KLS09] that the graph of non-separating discs in a torus with two marked points on the boundary is the Bass-Serre tree of the HNN-extension of the free group with two generators (the fundamental group of the torus with one marked point on the boundary) with respect to the boundary curve of the disc. In particular, it is hyperbolic. $\qquad\square$

9. Relation to the free factor graph

In this section we relate the electrified disc graph to the free factor graph of a free group F_g with $g \geq 3$ generators and show the proposition from the introduction.

We first define

Definition 9.1. The *free factor graph* $\mathcal{FF}$ is the metric graph whose vertices are conjugacy classes of non-trivial free factors of F_n. Two vertices x, y are connected by an edge of length one if there are representatives A of x, B of y and if there is a (possibly trivial) free factor C so that $F_g = A * B * C$.

The outer automorphism group $\text{Out}(F_g)$ of F_g acts on $\mathcal{FF}$ as a group of simplicial isometries. This action is coarsely transitive.

There is another version of the free factor graph which is defined as follows [KL09].

Definition 9.2. The *ellipticity graph* $\mathcal{ZG}$ is the bipartite graph whose vertex set is the union of the set of all conjugacy classes of nontrivial free splittings of F_g with the set of all nontrivial cyclic words of F_g. The vertices $A * B$ and w are connected by an edge of length one whenever w has a representative in A or B.

The group $\text{Out}(F_g)$ act on the ellipticity graph $\mathcal{ZG}$ as a group of simplicial automorphisms. This action is coarsely transitive.

An L-quasi-isometric embedding Φ of a metric space X into a metric space Y is called a *quasi-isometry* if for every $y \in Y$ there is some $x \in X$ with $d(\Phi(x), y) \leq c$. Kapovich and Lustig [KL09] observed that the ellipticity graph is $\text{Out}(F_g)$-equivariantly quasi-isometric to the free factor graph (compare also [BK10] for more properties of the ellipticity graph). Let $d_{\mathcal{F}}$ be the distance in the free factor graph. If we denote by $[A]$ the conjugacy class of a free factor A of F_g then we have

Lemma 9.3. *There is a number $\chi > 0$ with the following property. If $A, B \subset F_g$ are free factors which intersect non-trivially, then $d_{\mathcal{F}}([A], [B]) \leq \chi$.*

Let M be the connected sum of g copies of $S^2 \times S^1$. An embedded 2-sphere S in M is called *essential* if S does not bound a ball. The group $\text{Map}(M)$ of all isotopy classes of orientation preserving homeomorphisms of M projects onto $\text{Out}(F_g)$ with finite kernel. Each element in the kernel fixes each isotopy class of an essential sphere in M. Thus $\text{Out}(F_g)$ acts on the set of all isotopy classes of essential spheres in M.

In the next definition, we call an embedded oriented surface $S \subset M$ *nontrivial* if either S is an essential sphere or if the genus of S is at least one and the image of $\pi_1(S)$ under the homomorphism $\pi_1(S) \to \pi_1(M)$ induced by the inclusion $S \to M$ is nontrivial. For example, if $\gamma : S^1 \to M$ is an embedded non-contractible loop then the boundary of a small tubular neighborhood of γ is a nontrivial embedded torus.

Definition 9.4. The *electrified sphere graph* is the graph $\mathcal{ESG}$ whose vertices are essential spheres in M and where two such spheres S_1, S_2 are connected by an edge of length one if and only if up to isotopy there is a component of $M - S_1 - S_2$ which contains a nontrivial embedded oriented surface.

The group $\text{Out}(F_g)$ acts on $\mathcal{ESG}$ as a group of simplicial isometries.

If metric spaces X, Y admit isometric actions by groups G, H and if there is a homomorphism $\rho : G \to H$ and a map $\Phi : X \to Y$ such that for some $c > 0$

we have $d(\Phi(gx), \rho(g)\Phi(x)) \leq c$ for all $x \in X, g \in G$ then we call Φ *c-coarsely* $G - H$-*equivariant*. Note that this depends on the homomorphism ρ.

Proposition 9.5. *There is a coarsely* $\mathrm{Out}(F_g)$-*equivariant quasi-isometry*

$$\Phi : \mathcal{ESG} \to \mathcal{FF}.$$

Proof. Define a map Φ from the set of essential spheres in M (i.e. the set of vertices of $\mathcal{ESG}$) to the set of vertices of $\mathcal{FF}$ as follows.

Let $S \subset M$ be an essential sphere. If S is non-separating then cutting M open along S results in a bordered manifold which is homeomorphic to the connected sum of $n - 1$ copies of $S^2 \times S^1$ with $S^2 \times [0, 1]$.

Choose a basepoint $x \in M - S$. The subgroup $F(S)$ of $\pi_1(M; x)$ of all homotopy classes of based loops which are contained in $M - S$ is a free factor of $F_g = \pi_1(M; x)$ of corank one. The conjugacy class $\Phi(S)$ of this free factor only depends on the isotopy class of S.

If S is separating then choose a basepoint $x \in S$. By van Kampen's theorem, S defines a free splitting $\pi_1(S; x) = F_g = A * B$. Each of the groups A, B is a free factor of F_g. The distance in $\mathcal{FF}$ between the conjugacy classes of A, B equals one. Let $F(S)$ be one of these free factors and define $\Phi(S)$ to be the conjugacy class of $F(S)$. The thus defined map Φ from the set of vertices of $\mathcal{ESG}$ into the set of vertices of $\mathcal{FF}$ is coarsely $\mathrm{Out}(F_g)$-equivariant.

We claim that the map Φ is Lipschitz with Lipschitz constant $2\chi + 4$ where $\chi > 0$ is as in Lemma 9.3. To see this let $S, S' \subset M$ be two essential spheres of distance one in $\mathcal{ESG}$. Then there is a nontrivial surface S'' which is disjoint from $S \cup S'$.

Assume first that S'' is an essential sphere. By symmetry, it suffices to show that $d_{\mathcal{F}}(\Phi(S), \Phi(S'')) \leq \chi + 2$ where $\chi > 0$ is as in Lemma 9.3. However, in this case we can cut M open along $S \cup S''$ and obtain a bordered manifold. Since $n \geq 3$ by assumption, at least one component of $M - (S \cup S'')$ has non-trivial fundamental group. Thus by possibly replacing $\Phi(S), \Phi(S'')$ by conjugacy classes of free factors of distance one in $\mathcal{FF}$ we may assume that there are defining free factors for $\Phi(S), \Phi(S'')$ with non-trivial intersection. Thus the claim follows from Lemma 9.3.

On the other hand, if S'' is a surface of genus at least one then there is a component K of $M - (S \cup S')$ and a homotopically nontrivial loop in M which is entirely contained in K. But this means that for one of the at most two possible choices for the conjugacy classes $\Phi(S), \Phi(S')$ in the above definition there are representative free factors of F_g which intersect nontrivially. Then $d_{\mathcal{F}}(\Phi(S), \Phi(S')) \leq \chi + 2$ by the choice of χ and Lemma 9.3. This completes the proof that Φ is $2\chi + 4$-Lipschitz.

We are left with showing that Φ has a coarsely Lipschitz coarse inverse. To this end let $A \subset F_g$ be a free factor. Then a free basis $\alpha_1, \ldots, \alpha_k$ of A can be extended to a free basis $\alpha_1, \ldots, \alpha_n$ of F_g. Let B be the free factor of F_g generated by $\alpha_1, \ldots, \alpha_{n-1}$. The distance in $\mathcal{FF}$ between the conjugacy classes of A and B equals 2. As a consequence, the set of conjugacy classes of free factors of $\mathcal{FF}$ of corank one is 2-dense in $\mathcal{FF}$.

The group $\mathrm{Out}(F_g)$ acts transitively on conjugacy classes of free factors of F_g of corank one, and it acts transitively on essential non-separating spheres in M. Therefore there is an embedded non-separating sphere $S \subset M$ so that up to conjugation, B is the subgroup of F_g of all loops based at a point $x \in M - S$ which do not intersect S. In other words, $\Phi(S)$ is the conjugacy class of B. Associate to

the conjugacy class $[A]$ of A the sphere $\Lambda([A]) = S$. Then $d(\Phi(\Lambda[A]), [A]) \leq 2$ and hence Λ is a coarse inverse of Φ.

To show that Λ is coarsely Lipschitz it suffices to show the following. Assume that A, B are free splittings defined by spheres S, S'. If $A \cap B$ is non-trivial then $d(\Lambda[A], \Lambda[B]) \leq c$ where $c > 0$ is a universal constant. However, if $A \cap B$ is non-trivial then there is a component of $M - (S \cup S')$ which is not simply connected. Then $M - (S \cup S')$ contains an essential closed surface and hence $d_{\mathcal{F}}(\Lambda[A], \Lambda[B]) \leq 1$. This shows the proposition. $\square$

The action of the group of homeomorphisms of the handlebody H of genus g on $\pi_1(H) = F_g$ induces a surjective homomorphism $\rho : \mathrm{Map}(H) \to \mathrm{Out}(F_g)$. The following observation now completes the proof of the proposition from the introduction.

Lemma 9.6. *There is a coarsely* $\mathrm{Map}(H) - \mathrm{Out}(F_g)$-*equivariant coarsely Lipschitz surjection* $\Lambda : \mathcal{EDG} \to \mathcal{ESG}$.

Proof. Let M be the double of H, i.e. the manifold obtained from glueing two copies of H along the boundary with a boundary identification which maps boundaries of discs to boundaries of discs. For an essential disc D in H define $\Lambda(D)$ to be the double of D in M. By construction, Λ defines a coarsely Lipschitz map $\mathcal{EDG} \to \mathcal{ESG}$.

To show that Λ is coarsely surjective, simply note that $\mathrm{Map}(H)$ acts transitively on complete free factor decompositions of F_g, and each such decomposition can be realized by a reduced disc system (see [S00] for an explicit account). On the other hand, such decompositions are in bijection with reduced sphere systems. $\square$

Remark: Define the *sphere graph* $\mathcal{SG}$ to be the graph whose vertices are isotopy classes of essential spheres in M and where two such spheres are connected by an edge of length one if and only if they can be realized disjointly. The sphere graph is the analog for $\mathrm{Out}(F_g)$ of the disc graph for the handlebody group. Some of its properties have been investigated in [AS09].

The *edge splitting graph* of F_g is the graph whose vertices are conjugacy classes of non-trivial free splittings of F_g of the form $F_g = A * B$ where A, B are of rank at least one, and where two such splittings are connected by an edge of length one if and only if they have a common refinement. Sabalka and Savchuk [SS10] showed that the edge splitting graph is not hyperbolic. This graph can be identified with the graph of separating spheres [SS10].

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